M. Khovanov - Categorifying Quantum Groups

Note Title 4/22/2008

w/ A. Lauda

Goal: categorification of U_q^- for a simply-laced Cartan datum ... realize as $K_0(?)$

Γ graph, unoriented, no loops or multiple edges
$\underline{I}$ = Vertices (Γ)

Bilinear form $\cdot$ on $\mathbb{Z}[I]$

$$i \cdot j = \begin{cases} 2 & i=j \\ -1 & i \bullet\!\!-\!\!\bullet j \\ 0 & \overset{\bullet}{i} \quad \overset{\bullet}{j} \end{cases}$$

$'f$: free associative algebra / $\mathbb{Q}(q)$ on generators θ_i, $i \in I$,
$\mathbb{N}[I]$ - graded with $\deg \theta_i = i$

$$'f = \bigoplus_{\nu \in \mathbb{N}[I]} 'f_\nu$$

Equip $'f \otimes 'f$ with a twisted multiplication

$$(x_1 \otimes x_2)(x_1' \otimes x_2') = q^{-|x_1'| \cdot |x_2|} \, x_1 x_1' \otimes x_2 x_2'$$

Comultiplication $\Delta\theta_i = \theta_i \otimes 1 + 1 \otimes \theta_i$

Bilinear form on $'f$:

- $(1,1) = 1$
- $(\theta_i, \theta_j) = \delta_{ij} \frac{1}{1-q^2}$
- $(x, yy') = (\Delta x, y \otimes y')$
- $(xx', y) = (x \otimes x', \Delta y)$

I = kernel of $(,)$ is a 2-sided ideal in $'f$

$$f := {}'f/I = \bigoplus_{\mathbb{N}[I]} f_\nu$$

$$\left.\begin{array}{ll} \theta_i\theta_j - \theta_j\theta_i & \overset{i}{\bullet} \quad \overset{j}{\bullet} \\ \theta_i\theta_j\theta_i - \theta_i^{(2)}\theta_j - \theta_j\theta_i^{(2)} & \overset{i}{\bullet}\!\!-\!\!\!-\!\!\overset{j}{\bullet} \end{array}\right\} \in I$$

$\left(\text{Here } \theta_i^{(n)} = \frac{\theta_i^n}{[n]!} \right)$

Theorem (q-Gabber-Kac) I is generated as a 2-sided ideal by these elements.

f is a twisted bialgebra

Integral form $f_{\mathcal{A}} \subset f$:

$\mathbb{Z}[q,q^{-1}]$ - subalgebra generated by all divided powers

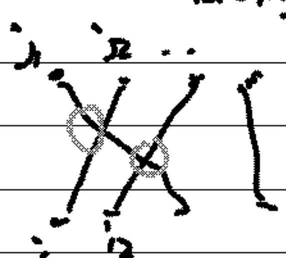
$\theta_i^{(n)}$ $n \geq 0$ $i \in I$

<u>Geometric Interpretation</u> of $(,)$

Let $seq(\nu)$ = all sequences $i_1 i_2 \dots i_m$ of vertices in I with $\nu = i_1 + i_2 + \dots + i_m$, $\nu \in \mathbb{N}[I]$

Let $\theta_{\underline{i}} = \theta_{i_1} \theta_{i_2} \cdots \theta_{i_m}$ $\underline{i} \in seq(\nu)$

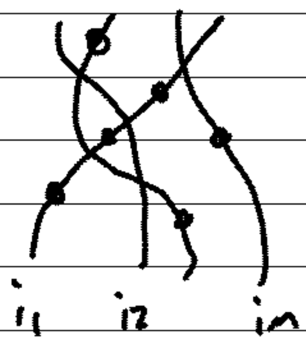
$$(\theta_{\underline{i}}, \theta_{\underline{j}}) = \sum_{\substack{\text{permutations} \\ \text{taking } \underline{i} \text{ to } \underline{j}}} q^{-\sum_{i \times j} i \cdot j} \left(\frac{1}{1-q^2}\right)^m$$



... sum over intersections

Now replace $\mathbb{Z}$ by a field $\mathbb{k}$

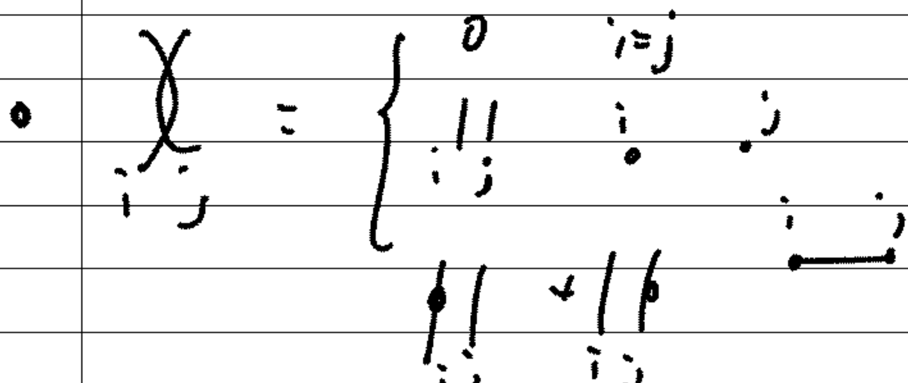
Draw pictures in the plane where strands carry dots

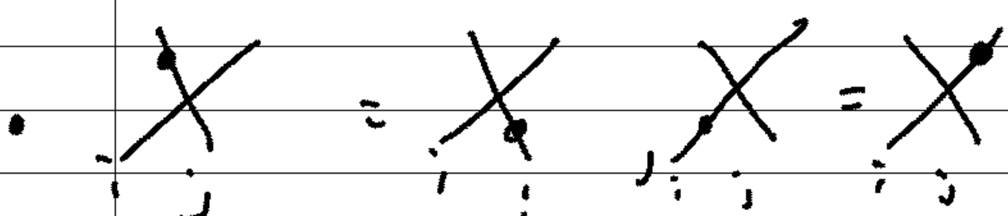


(up to obvious isotopies)

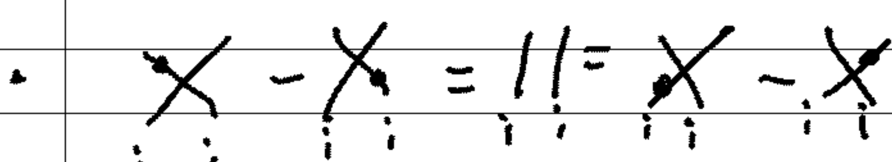
$\nu = i_1 + \dots + i_n$ fixed

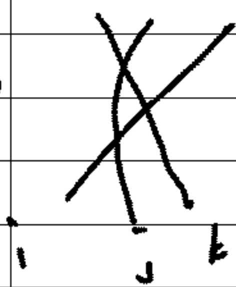
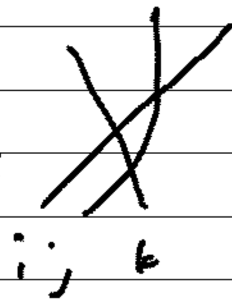
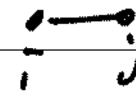
Ring $R(\nu)$ generated by these pictures with relations

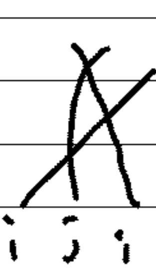
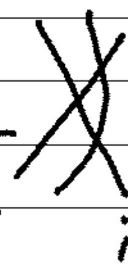


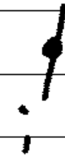


if $i \neq j$



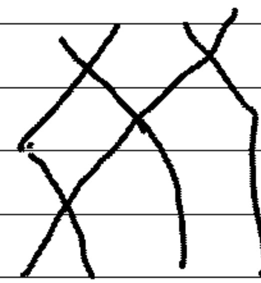
•  =  unless $i=k$ & 

•  −  = 

• deg  = 2 , deg  = $-i\cdot j$

$\Longrightarrow$ any diagram is equivalent to a diagram (possibly with many more dots) which is a sum of reduced presentations of permutations, & all dots are at top of diagram

Fix a reduced word w' for each $w \in S_n$

$w' \longmapsto \tilde{w}$ picture 

So any element can be written as $\sum_{w \in S_n} \underbrace{\left(\sum \right)}_{\text{dots at top}} \tilde{w}_{i_1 \dots i_m}$ $\underline{i}$ given sequence at bottom

$R(\nu) = \bigoplus_{Seq(\nu)^2} {}_{\underline{j}}R(\nu)_{\underline{i}}$ $\underline{i}$ bottom, $\underline{j}$ top

$P_{\underline{i}} = \bigoplus_{\underline{j}} {}_{\underline{j}}R_{\underline{i}}$ left projective module

${}_{\underline{j}}P = \bigoplus_{\underline{i}} {}_{\underline{j}}R_{\underline{i}}$ right projective module

} graded

$\mathrm{gr}\dim {}_{\underline{j}}R_{\underline{i}}(\nu) \leq (\theta_i, \theta_j)$

($\frac{1}{(1-q^2)^m}$ term accounts for dots)

... actually want an equality on the graded dimension..

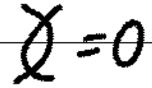
Example: single vertex $\bullet^i$

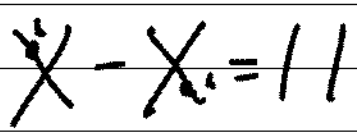
$R(n\, i) \simeq NH_n$ Nil Hecke algebra

= Endos of $k[x_1, \dots x_n]$

generated by $x_k \cdot \_$ & divided differences

$$\partial_k(f) = \frac{f - f^{s_k}}{x_k - x_{k+1}}$$

$\partial_k^2 = 0 \iff$  $= 0$

 is commutator of $x_k \cdot \_$ & ∂_k giving identity

+ Yang-Baxter $\partial_k \partial_{k+1} \partial_k = \partial_{k+1} \partial_k \partial_{k+1}$

$$NH_n \simeq Mat(n!, k[x_1, \dots x_n]^{S_n})$$

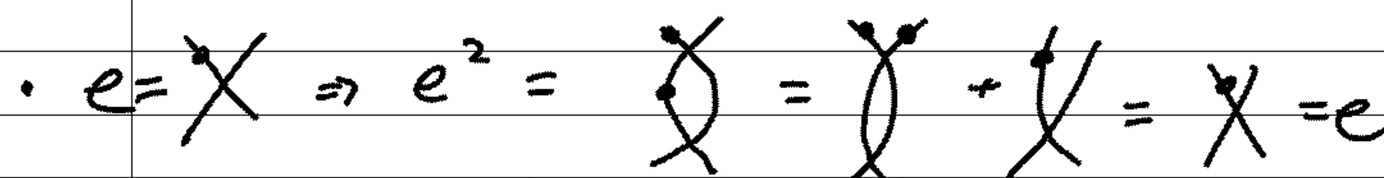
-- all operators commute with symmetric polynomials.
get matrix algebra of rank $n!$ over this subalgebra.

$NH_m \cong \bigoplus_{m!} P_m$ $m!$ copies of the indecomposable projective

- start in deg $-\frac{m(m-1)}{2}$ for balance.

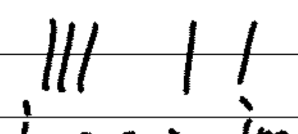
- Divided powers will correspond to passing to indecomposable summands!

$$\begin{array}{c} \theta_i^{\,m} \longrightarrow \theta_i^{(m)} \\ \Longleftrightarrow \\ NH_m \longrightarrow P_m \end{array}$$



idempotent

Note for all $\underline{i}$ have idempotent

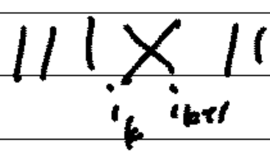
$1_{\underline{i}} =$  $\& \; 1 = \sum_{\underline{i} \in Seq(\nu)} 1_{\underline{i}}$

$R(\nu)$ acts faithfully on Pol_ν multi-polynomials

$\mathrm{Pol}_\nu = \bigoplus_{Seq(\nu)} \mathrm{Pol}_{\underline{i}}$, $\mathrm{Pol}_{\underline{i}} = k[X_1 \dots X_m]$

$1_{\underline{i}}$ acts as identity of $\mathrm{Pol}_{\underline{i}}$ & 0 elsewhere

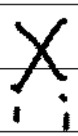
$\mathrm{Pol}_{\underline{i}}$ as $X_k \cdot \_$

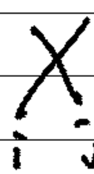


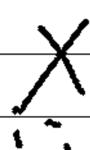
$\underline{j} = s_k \underline{i}$ $\quad \mathrm{Pol}_{\underline{j}}$

$\underline{i}$ $\quad \mathrm{Pol}_{\underline{i}}$

: by cases

-  : apply divided difference from $\mathrm{Pol}_{\underline{i}}$ to $\mathrm{Pol}_{\underline{i}}$

-  i j : just transpose X_k & X_{k+1}

-  with $i - j$: choose orientation of graph

Γ

do transposition if $i \leftarrow j$

& $f \mapsto (X_k + X_{k+1}) \cdot f^{s_k}$ if $i \rightarrow j$

Check relations ... needs some shifts to make the representation graded.

Then find that our $\tilde{\omega}$ give a basis in $R(v)$

$$\Longrightarrow \boxed{\mathrm{gdim}\ {}_{\underline{i}}R(v)_{\underline{i}} = (\theta_i, \theta_i)}$$

(This rep should be equiv. cohomology of θ fiber in Lusztig moment map projection for quiver flag variety, ${}_{\underline{i}}R(v)_{\underline{i}}$ should be part of convolution algebra ...)

... pictorial restatement of Lusztig construction.

$$Z(R(v)) = \bigotimes_i \mathbb{Z}[x_1, \dots x_{v_i}]^{S_{v_i}}$$

$$\simeq H^*_{\prod GL(v_i)}(\cdot)$$

Want to consider $R = \bigoplus_{v \in \mathbb{N}[I]} R(v)$

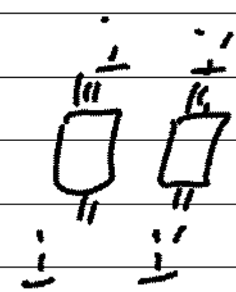
$K_0(R) := \bigoplus_{\nu} K_0(R(\nu))$ K-group of f.g. graded left projective $R(\nu)$-modules

a $\mathbb{Z}[q,q^{-1}]$-module

Grading is bounded below, R noetherian, fin many indecomposables all ν

$\rightsquigarrow$ $K_0(R)$ has a basis $[P_b]$ indecomposable projectives $b \in B(\nu)$ parametrizing set

<u>$K_0(R)$ is a twisted bialgebra:</u>



$R(\nu) \otimes R(\nu') \subset R(\nu+\nu')$

(nonunital:

$1 \otimes 1 \longmapsto 1_{\nu,\nu'}$ idempotent)

$\Rightarrow$ induction & restriction functors

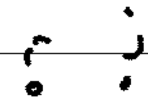
Ind takes projectives to projectives

$$K_0(R(\nu)) \underset{\mathbb{Z}[q,q^{-1}]}{\otimes} K_0(R(\nu')) \underset{[\mathrm{Res}]}{\overset{[\mathrm{Ind}]}{\rightleftarrows}} K_0(R(\nu+\nu'))$$

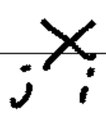
$\boxed{\gamma : f_{\mathcal{A}} \longrightarrow K_0(R)}$ map of twisted bialgebras

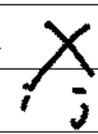
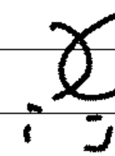
$\gamma : \theta_{\underline{i}} = \theta_{i_1} \cdots \theta_{i_m} \longmapsto P_{\underline{i}}$ corresponding projective

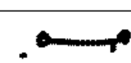
For divided powers need to act by a suitable idempotent on $P_{\underline{i}}$ to get a smaller module.

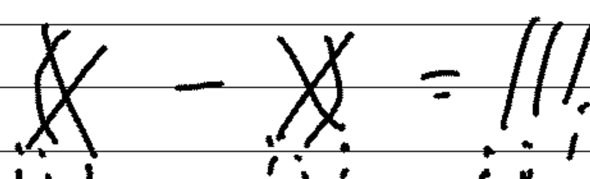
Relations: • $\theta_i \theta_j = \theta_j \theta_i$ 

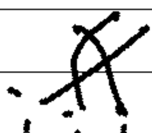
$P_{\dots ij \dots} \cong P_{\dots ji \dots}$

via multiplication by ,

inverse given by  (composition  $= || = \mathrm{id}$)

• $\theta_i \theta_j \theta_i = \theta_i^{(2)} \theta_j + \theta_j \theta_i^{(2)}$ 

look carefully at 

... find  is an idempotent

$\rightsquigarrow$ find idempotents in $P_{\dots iji \dots}$
projecting to $P_{\dots i^{(2)} j \dots}$ and $P_{\dots j i^{(2)} \dots}$

...

$\gamma : f_{\mathcal{A}} \longrightarrow K_0(R)$ is injective since
it takes the bilinear form on $f_{\mathcal{A}}$ to
form on K_0 $\quad ([P],[Q]) = \text{grdim}\,(P^{\psi} \otimes_{R(\nu)} Q)$
ψ = reflection of diagram top $\leftrightarrow$ bottom
... in particular $\ _{\underline{j}}R(\nu)_{\underline{i}} \simeq P_{\underline{j}}^{\psi} \otimes P_{\underline{i}}$

γ is surjective: use Kleshchev-Grojnowski-Vazirani
machinery ... work in basis of simples
& ind/res functors on them... socle
of restriction is irreducible $\rightsquigarrow$ crystal
graph structure on simples.
(Ch. 5 of Kleshchev book)

- Generalize to nonsimply laced case:
$i \cdot i \in \{2, 4, \dots\}$ ---> change degrees $\leftrightarrow$ dots

[relations $f \mapsto (X_i^{\bullet} + X_{i+1}^{\bullet}) f$

set powers ...]

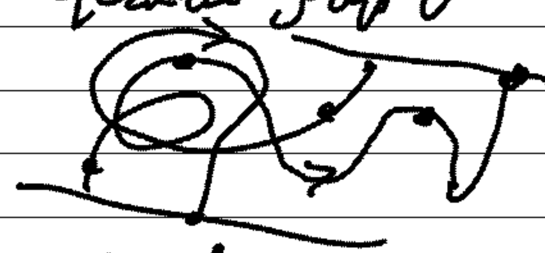
- add more parameters to theory in non simply laced case.

This works over any field k $\Rightarrow$

basis of $f_{\mathcal{A}}$ corresponding to projectives $[P_b]$, which should generalize Lusztig's canonical basis (corresponding to $k = \mathbb{C}$)

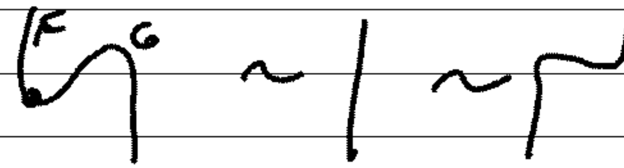
$$\mathrm{Hom}(P_b, P_b) = \underset{\deg 0}{k} \oplus \text{higher deg terms.}$$

Expect categorification of full quantum group $\dot{U}$ by looking at oriented strands



not just strands from bottom to top

(A. Lauda for $\dot{U}(sl_2)$)

Biadjointness : 

Dots $\longleftrightarrow$ generators of $H^*(\mathbb{CP}^\infty)$