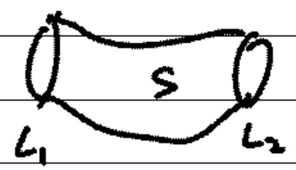
Mikhail Khovanov - Link Homology & Categorification

Note Title 1/16/2008

Link homology: $L \subset S^3 \rightsquigarrow H(L)$



link cobordism: $H(S): H(L_1) \to H(L_2)$

... simple version of 4d TQFT:

only see cobordisms $S \subset S^3 \times [0,1]$.

Extend to tangles!

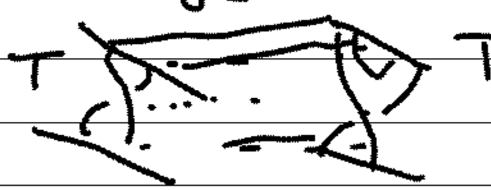
2-category of tangle cobordisms

objects: n-tuples of points in plane

1-morphisms: tangles

$\subset \mathbb{R}^2 \times [0,1]$

2-morphisms: tangle cobordisms $S \subset \mathbb{R}^2 \times [0,1]^2$



presented boundary

Consider 2-functors

n-tangles $\longrightarrow$ C_n = complexes of modules over a ring H^n

cobordisms $\longrightarrow$ functors

[everything oriented]

Case of trivial link 
with simplest cobordisms  etc
give 2d TQFT F

$F: \emptyset \rightsquigarrow R$ commutative ring

$F: O \rightsquigarrow A$ free R-module, which inherits structure of a commutative Frobenius R-algebra

$$A \cong A^* = \mathrm{Hom}_R(A, R)$$

Let B^n = crossingless matchings of $2n$ points

n disjoint arcs in bottom half plane

$$|B^n| = \frac{1}{n+1}\binom{2n}{n}$$

Given $a, b \in B^n$ a  b   $w(b)$ reflected version

$\Rightarrow$ glue to get closed 1-manifold

$w(b)\,a$ 

$$\Rightarrow \quad F(w(b)a) \cong A^{\otimes k} \qquad k = \#\text{ cycles}$$

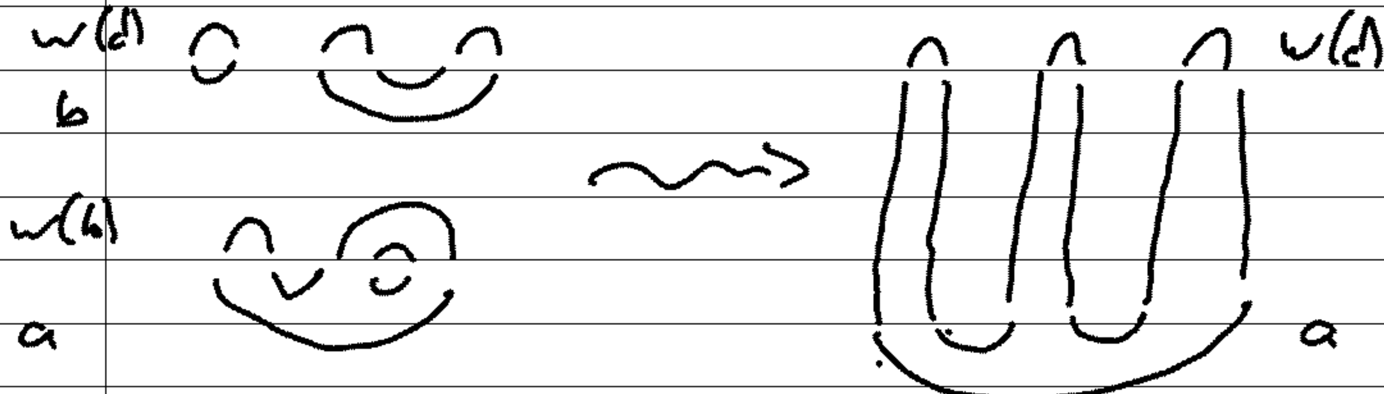
Let $H^n = \bigoplus_{a,b \in B^n} F(w(b)a)$

Claim: H^n is an associative unital

R-algebra : product

$F(w(d)c) \otimes F(w(b)a)$ product is zero if $c \neq b$

$$F(w(d)b) \otimes F(w(b)a) \longrightarrow F(w(d)a)$$

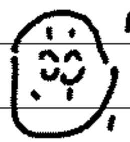


apply cobordism made of n cobordisms of form $\overset{\smile}{\frown} \longrightarrow ||$

$\rightsquigarrow$ gives  morphism from input to output as desired.

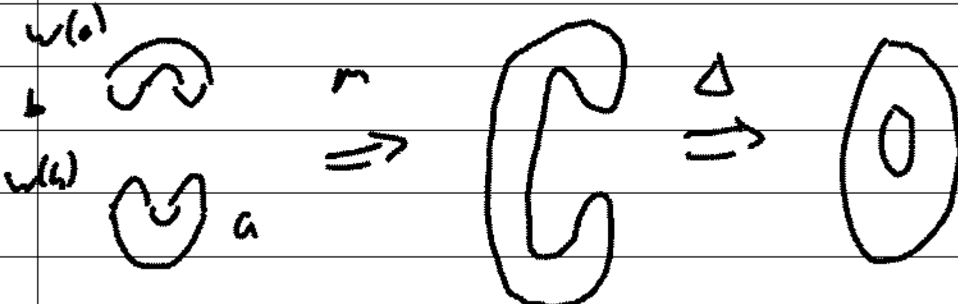
Associativity: cobordisms we get are diffeomorphic.

$a \in B^n$ $\quad F(w(a)a) \simeq A^{\otimes n} \ni 1^{\otimes n} =: 1_a$



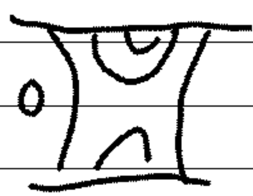
$x \in F(w(b)a)$

$\Rightarrow x 1_a = x$, etc.



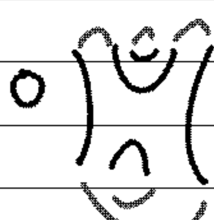
$$A^{\otimes 2} \xrightarrow{m} A \xrightarrow{\Delta} A^{\otimes 2}$$

Given H^n $\rightsquigarrow$ produce invariants of flat tangles & their cobordisms : flat tangles live in plane

T 

(proper embedding of 1-manifold into $\mathbb{R}\times[0,1]$) require even # ends

to T assign (H^m, H^n)-bimodule



$$F(T) = \bigoplus_{\substack{a \in B^n \\ b \in B^m}} F(w(b)Ta)$$

Sum over all ways to close up T with $a \in B^n$, $b \in B^n$ get bimodule

$$F(w(b)T_a) \otimes F(w(a)c) \longrightarrow F(w(b)T_c)$$

etc.

e.g. $T = ||||||$ $F(T) = H^n$.

$a \in B^n \rightsquigarrow P_a = F(a)$ left H^n-module

$$= \bigoplus_b F(w(b)\, a)$$

projective H^n-module.

$H^n = \bigoplus_{a \in B^n} P_a$: complete list of indecomposable H^n-modules.

Composition of tangles $\Rightarrow$ tensor of bimodules
cobordism of tangles $\Rightarrow$ maps of bimodules
...

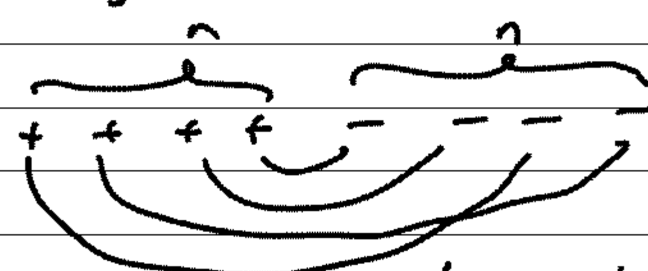
$\Rightarrow$ F 2-functor from tangle cobordisms
to the 2-category of bimodule homomorphisms
objects = rings 1-morphisms = bimodules ....

This invariant cares about the topology of the cobordisms but not their knotted embeddings in $\mathbb{R}^3$....

Different picture: forget about $\mathbb{R}^2, \mathbb{R}^3$ & embeddings:
work with oriented 1-manifolds with boundary & 2-manifolds with corners

$H^n \rightsquigarrow$ bigger version: look at all oriented arcs relating two oriented 0-manifolds

$\widetilde{B}^n$:



look at all couplings, $n!$ such

$\Rightarrow$ $\widetilde{H}^n$ bigger ring

$\to$ minimal extension of a 2d TQFT (R,A) to a 2d TQFT with corners (extended TQFT)

... restrict to zero manifolds with same number of + & − :

2-category:
balanced oriented 0-manifolds — rings $\widetilde{F}^n$
oriented cobordisms
cobordisms of cobordisms

3d: bimodules $\rightsquigarrow$ 4d: complexes of bimodules

3d TQFTs can be interesting with semisimple or abelian categories

4d TQFTs: expect to need triangulated categories.

closed surface $\rightsquigarrow$ triangulated category

3d cobordism $\rightsquigarrow$ exact functor

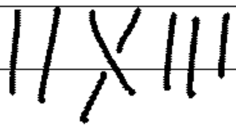
4d cobordism $\rightsquigarrow$ natural transformation of functors

Why? S surface, diff $S \circlearrowright$ category $C(S)$

If S semisimple $\Rightarrow$ $MCG(S) \to$ permutations of simple objects, no interesting such...

Tangle $T \rightsquigarrow$ complex of (m,n) bimodules:

write plane projection of T



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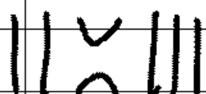
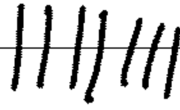


D_0



D_1 flat tangles

Have a saddle point cobordism $D_0 \overset{S}{\rightsquigarrow} D_1$

So can assign to D the complex

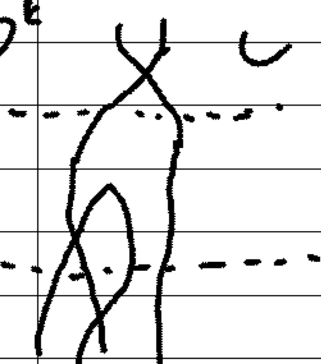
$$F(D_0) \xrightarrow{F(S)} F(D_1)$$

To an arbitrary D, decompose into diagrams with at most one crossing

D^k

D^2

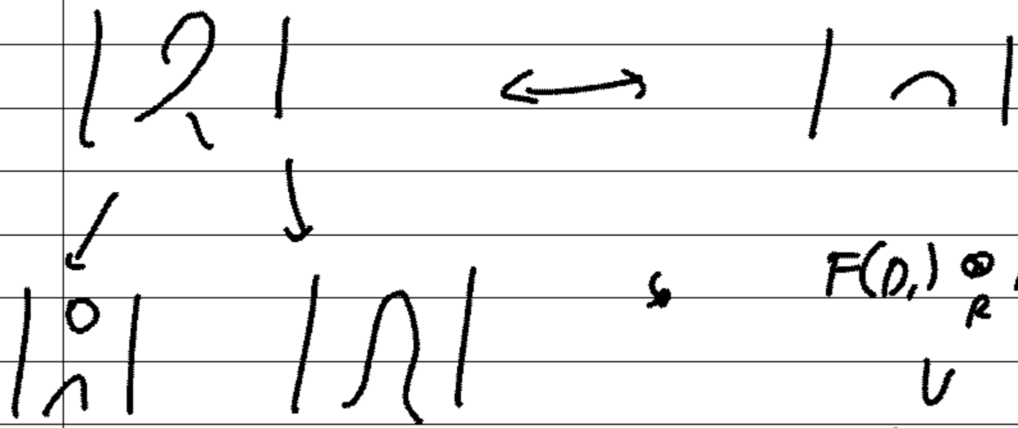
D^1



$$\rightsquigarrow F(D) = F(D^k) \otimes_H \cdots \otimes F(D^1)$$

Need to check if this is independent of the decomposition up to quism..

$\rightsquigarrow$ need $A/R \cong R$ as R-mods



D_0 $\qquad$ D_1

so

$$\begin{array}{ccc} F(D_1)\otimes_R A & \longrightarrow & F(D_0) \\ \cup & & \parallel \\ F(D_1)\otimes_R R & \cong & F(D_1) \end{array}$$

$$\Rightarrow F(D_1)\otimes_R A/R \cong F(D)$$

— So can ask $A = R\cdot 1 \oplus R\cdot X$

get invariance under

get $F(D)$ an invariant of underlying tangle only.

Two interesting cases

1. $R = \mathbb{Z} = H^*(\cdot, \mathbb{Z})$ $\qquad A = \mathbb{Z}[x]/x^2 = H^*(S^2, \mathbb{Z})$

2. $R = \mathbb{Q}[t] = H^*_{SU_2}(\cdot, \mathbb{Q})$

$A = \mathbb{Q}[x]/x^2 = t \quad = H^*_{SU_2}(S^2, \mathbb{Q})$

R, A are graded $\quad \deg x = 2 \quad \deg t = 4.$

$\rightsquigarrow$ make all rings H^n graded, $F(S)$ homogeneous ....

... $F(D)$ complex of graded (m,n)-bimodules

Construction extends to tangle cobordisms in $\mathbb{R}^4$:

Encode diagramatically in standard moves
(Reidemeister, creation & annihilation of
a circle $\cdot \longleftrightarrow O$, saddle point $)(\longleftrightarrow \genfrac{}{}{0pt}{}{\smile}{\frown}$)

$\cdot \to O$ unit $O \longrightarrow \cdot$ trace

<u>Theorem</u> $\pm F(S)$ is an invariant of S

... invariance under movie moves (18 such)

Braid $D \rightsquigarrow F(D)$ is an <u>invertible</u> complex

$F(D^{-1}) \otimes F(D) \simeq H^n$

$\mathrm{Hom}(F(D), F(D)) \simeq \mathrm{Hom}(H^n, H^n) = Z(H^n)$

Degree zero part is only ± 1 .....

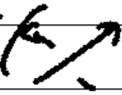
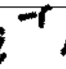
so Reidemeister movie moves give invariance up to sign.

$Z(H^n) \simeq H^*$(Springer fiber for partition (n,n))

...irred components are all products of S^2's, labeled by B^n.

(Scott Morrison, Kevin Walker: get rid of signs by paying careful attention to orientations)

If our tangle is a link L
$\Rightarrow$ $F(L)$ is just a complex of graded abelian groups
char polynomial gives the Jones polynomial!

$q^2 J($  $) - q^{-2} J($  $) = (q - q^{-1}) J($  $)$

$$J(\theta) = q + q^{-1} \qquad J(L) = \chi(F(L))$$