M. Khovanov - Link homology II: Matrix Factorization

Note Title 1/23/2008

Last time: A commutative R-algebra

$$A = R\cdot 1 \oplus R\cdot X \quad (rk_R A = 2)$$

e.g. $R = \mathbb{Z}[h,t]$, $\deg h = 2$ $\deg t = 4$

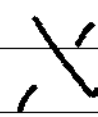
$$= H^*_{U(2)}(pt)$$

$$A = R[x]/(x^2 - hx - t) \quad \deg x = 2$$

$$= H^*_{U(2)}(S^2)$$

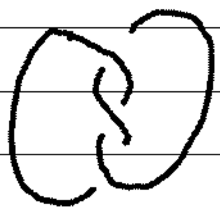
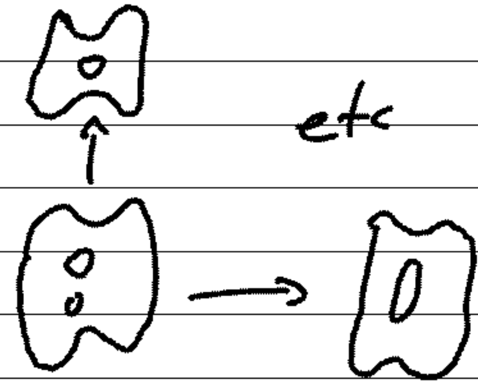
with trace $\varepsilon(x) = 1$, $\varepsilon(1) = 0$
$[\varepsilon : A \to R]$

$\Rightarrow$ algebras H^n & bimodule complexes
$F(T)$ for tangles T
$\rightsquigarrow$ bigraded homology invariants.

For a link can directly define the homology:
for every crossing  resolve in two ways

n crossings $\Rightarrow$ 2^n resolutions,
put as vertices of an n-dimensional cube

Each vertex $\longleftrightarrow$ collection of circles

 $\Rightarrow$  etc

$\Rightarrow$

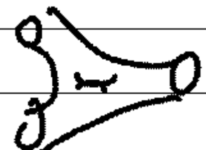
$$\begin{tikzcd} A^{\otimes 2} & \\ A^{\otimes 3} \arrow[u] \arrow[r] & A^{\otimes 2} \end{tikzcd}$$

etc

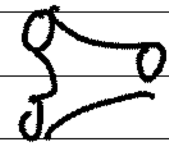
commutative diagram coming from TQFT

F applied to the above 1-manifolds & the cobordisms between them coming from edges.

Shift grading of A so 1 sits in deg. -1

$\Rightarrow$ $F($ S =  $) : F(\partial_0 S) \to F(\partial_1 S)$

will have degree $-\chi(S)$

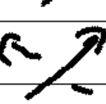
e.g  has degree 1 $\longleftrightarrow \chi = -1$

$$\left[\begin{array}{ccc} 1\otimes 1 & \longrightarrow & 1 \\ \deg -2 & \longrightarrow & -1 \end{array}\right]$$

In above commutative diagram each edge cobordism comes from the above S $\Rightarrow$ has degree 1. we'll shift grading on targets of arrows by 1 so all edges have degree 0

$$\begin{array}{ccc} A^{\otimes 2}\{-1\} & \longrightarrow & \\ \uparrow & \text{etc} & \uparrow \\ A^{\otimes 3} & \longrightarrow & A^{\otimes 2}\{-1\} \end{array}$$

$\Rightarrow$ commutative cube of graded R-modules
collapse grading $\Rightarrow$ complex $\overline{F}(D)$ of graded R-modules, differential has degree 0

Normalization on orientations of link diagram D:
let $x(D)$ = # negative crossings 
$y(D)$ = # positive crossings 

Shift $\bar{F}(D)\{2x(D)-y(D)\}[x(D)] =: F(D)$

$$\underset{-x(D)}{0 \to A^{\otimes k}} \to \cdots\cdots \to \underset{y(D)}{A^{\otimes \ell}} \to 0$$

<u>Theorem</u> Up to chain homotopy equivalence, $F(D)$ depends only on the underlying oriented link

$H(D) = H(F(D))$ homology: bigraded R-modules

& $\chi(H(D)) = \text{Jones}(L)$ Jones polynomial.

The width of the homology $\leq$ # crossings so gives lower bound on # crossings from homology

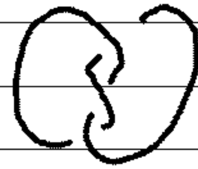
J. Rasmussen: let $h=0$ $R = \mathbb{Q}[t]$

$A = \mathbb{Q}[x]$ $x^2 = t$

$H_t(L) \supset \mathrm{Tor}(L) = \{v : t^N v = 0 \quad N \gg 0\}$

$0 \to \mathrm{Tor}(L) \to H_t(L) \to H'(L) \to 0$

free $\mathbb{Q}[t]$-module

e.g for trefoil 

H_t (trefoil) =

	0	1	2	3
complex	A		$A \xrightarrow{x} A$	
homology	$\mathbb{Q}[t] \oplus \mathbb{Q}[t]$ free			$\mathbb{Q}$ torsion

$\forall$ knot $H^1(L) \simeq A\{-s(L)-1\}$ for for some even integer $\underbrace{s(L)}$: rest of homology is torsion. Rasmussen invariant

Set $t=1$ $x^2=1 \Rightarrow A$ is semisimple.

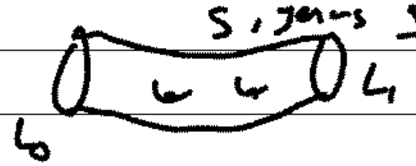
write $\alpha, \beta = \frac{x \pm 1}{2}$

product $\alpha \otimes \alpha \xrightarrow[m]{} \alpha$

$\alpha \xrightarrow[\Delta]{} 2\alpha \otimes \alpha$ Same for β

For a link L with k components, $H_{t=1}(L)$ has rank 2^k, with degrees $\leftrightarrow$ linking numbers. ($t=1$ kills all torsion

H' is functorial under cobordisms of links



$1 \quad x \quad x^2 = t \dashrightarrow 1 \quad x \quad x^2 = 1$

$$A\{-s(L_0)-1\} \longrightarrow A\{-s(L_1)-1\}$$

$0 \neq H'(S)$:

$\deg = -\chi(S) = 2g$: gives lower bound on genus of cobordism:

$$g \geq \left| \frac{s(L_0) - s(L_1)}{2} \right|$$

For positive knots the Rasmussen invariant is easy to compute: all crossings $\nearrow$, complex starts in degree zero

$s(L) = \frac{n+1-c}{2}$ $\quad c = \#$ circles when resolved)(

Let $g_4(L)$ = minimal genus of a smooth surface in D^4 bounding L $\Rightarrow$

$$g_4(L) \geq \left| \frac{s(L)}{2} \right|$$

$g(L)$ = Seifert genus = genus of spanning surface in S^3

$\rightsquigarrow$ find for positive links that

$g(L) = g_4(L)$ - conjecture of Milnor

resolved by Kronheimer-Mrowka first using gauge theory — above gives algebraic proof.

Note: Chern-Simons theory gives Jones polynomial with $q^N = 1$ ... one needs to replace this with a formal parameter q then categorify as a grading:

$$\begin{array}{ccc} q^N = 1 & \rightsquigarrow & q \text{ formal parameter} \\ \Big\downarrow & & \Big\downarrow \text{categorify} \\ ?? & \rightsquigarrow & \text{extra grading} \end{array}$$

Cyclically graded not enough to capture this: $q = e^{2\pi i/N}$, need condition $\varphi_N(q) = 0$ (cyclotomic polynomial) to capture q as complex number!

need $\mathcal{C}$ tensor triangulated category
with $K(\mathcal{C}) = \mathbb{Z}[q]/\varphi_N(q)$.

For $N=p$ prime can take

$B = k[y]/y^p$ where char.$(k)=p$

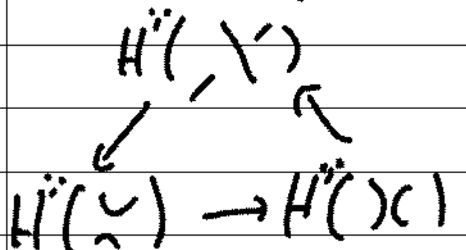
$\deg y = 1$ Hopf algebra with

$$\Delta y = y \otimes 1 + 1 \otimes y$$

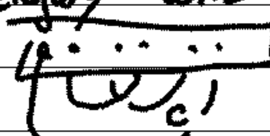
$\mathcal{C}$ = stable category of graded B-modules
is tensor triangulated
& $K(\mathcal{C}) = \mathbb{Z}[q]/\varphi_p(q)$.

Don't know constructions for N composite.

Historically first got H for <u>links</u>, then
want to extend to tangles: use triangles



To extend to tangles:
need category with object for
each



Using above triangles can reduce problem to defining objects of a category for every crossingless matchings $\rightsquigarrow$ hence origin of H^n

Gives procedure to try to extend TQFTs to lower dimension:

start with 4d TQFT, want to extend to surfaces $S \longrightarrow$ category $F(S)$

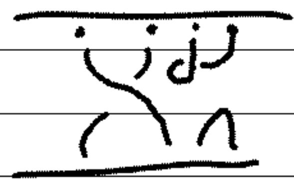
look at all 3-manifolds M bounding S, try to generate $F(S)$ as an Ext algebra by having enough M_i with $\partial M_i = S$

& Ext algebra $\mathrm{Hom}([M_i],[M_j])$

$$= F(M_i \underset{S}{\cup} M_j)$$

$\rightsquigarrow$ try $F(S) = D^b\left(\bigoplus_{i,j \in I} F(M_i \cup M_j)\text{-mod}\right)$

Example: $U_q(\mathfrak{g})$, given tangle colored by reps of $U_q(\mathfrak{g})$ have Reshetikhin-Turaev invariants



Can look only at <u>invariant</u> spaces of tensor products $\longleftrightarrow$ diagrams

Invariant of links

$$q^n P_n(\nwarrow\!\!\!\!\nearrow) - q^{-n} P_n(\nwarrow\!\!\!\!\nearrow) = (q-q^{-1}) P_n(\,)(\,)$$

$\rightsquigarrow$ $U_q(\mathfrak{gl}_n)$ invariant (for $n>0$)

... all strands labelled by standard rep.

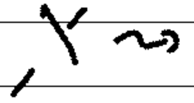
$n=0 \iff U_q(\mathfrak{gl}(1|1))$ invariant

More generally $U_q(\mathfrak{gl}(n|m)) \longrightarrow q^{n-m}$ power above.

(depends only on $n-m$)

$n=1$ get trivial invariant

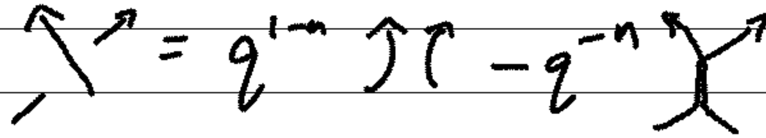
How do we categorify these invariants?

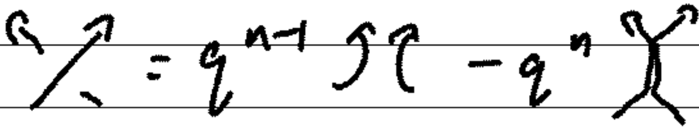
for $n=2$ we just used ,)(

& corresponding relation on Jones

(we're ignoring orientations now).

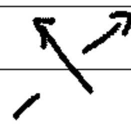
For general n use





(crossing reversal $\Longleftrightarrow$ $(q \leftrightarrow q^{-1})$)

New pictures: $V \xleftarrow{\mathrm{id}} V$ $V =$ fundamental rep of $U_q\mathfrak{gl}_n$

 $V^{\otimes 2} \to V^{\otimes 2}$  : R-matrix

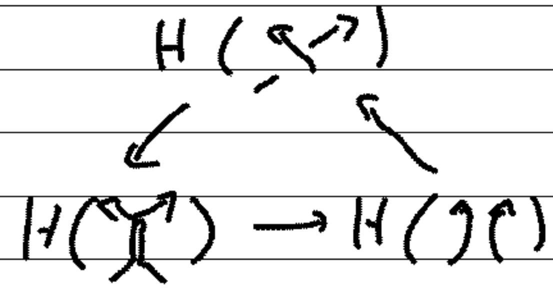
 $V^{\otimes 2} \to \Lambda^2_q V \to V^{\otimes 2}$: $\mathrm{Hom}_{U_q}(V^{\otimes 2}, V^{\otimes 2})$ is two dimensional & above give basis

When $n=2$ $\Lambda^2 V$ = trivial so $= \uparrow\uparrow$

& set $= \quad + \;)($

but otherwise have a new intertwiner ....

So would like triangle



& similar for .

Given projection of a link can resolve

& write in terms of intertwiners...

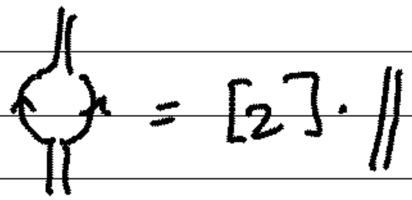
or use Reshetikhin-Turaev state sum formula:

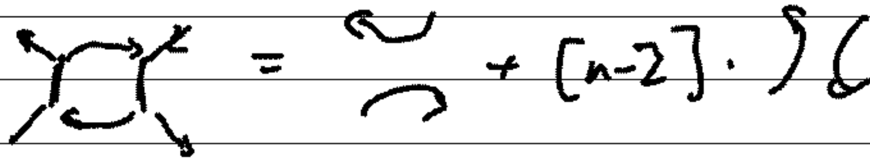
graph $\Gamma \longrightarrow P_n(\Gamma) \in \mathbb{Z}[q, q^{-1}]$

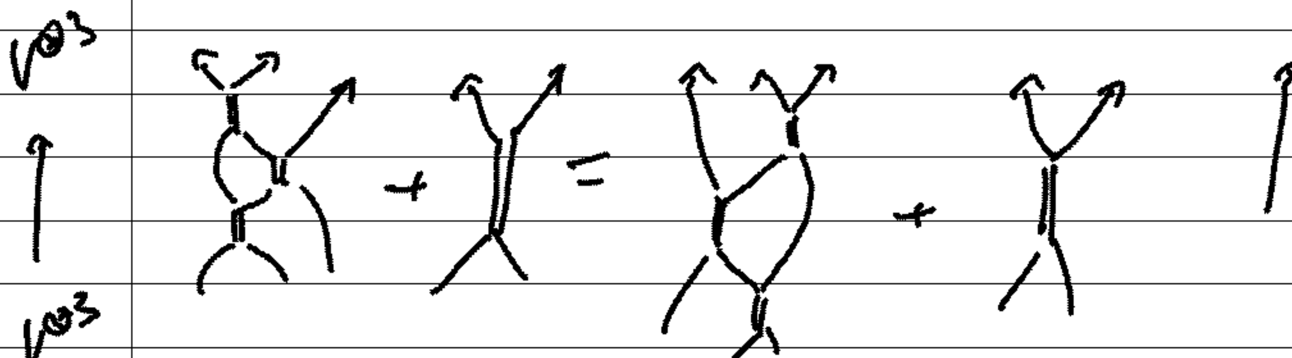
with rules

$$\bigcirc = [n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + \dots + q^{1-n}$$

$$= [n-1] \cdot \uparrow$$







– relation between intertwiners for $U_q\mathfrak{sl}_n$:

Get invariant of closed planar graphs of this kind

— or invariant of graphs in $\mathbb{R}^3$ with representations on edges & intertwiners on vertices.

To categorify $P_n(\Gamma) \rightsquigarrow H(\Gamma) = \bigoplus_{i\in\mathbb{Z}} H_i(\Gamma)$

$P_n(\Gamma) = \sum q^i \operatorname{rk} H_i(\Gamma)$, Γ planar

$$P_n(\bigcirc) = [n] = \mathrm{grdim}\, H^*(\mathbb{CP}^{n-1}, \mathbb{Q})\{1-n\}$$

So can take $H(\bigcirc) = \mathbb{Q}[x]/x^n \quad \deg x = 2$
(up to shift)

$$P_n(\bigcirc\!\!\bigcirc) = P_n(\bigcirc)[n-1] = [n-1][n]$$
$$= \mathrm{grdim}\, H^*(\text{flags } L_{n-2} \subset L_{n-1} \subset \mathbb{C}^n)$$

. . . .

To graph with boundary want object of a triangulated category $\rightsquigarrow$

<u>Matrix factorizations</u>

"complexes" with $D^2 = f$

$R = \mathbb{Q}[x_1, \ldots x_n] \ni f$ potential, $f \in \mathfrak{m}^2$ to make things interesting
$\cup$
$\mathfrak{m} = (x_1, \ldots x_n)$

f is nondegenerate if $\frac{\partial f}{\partial x_1}, \ldots \frac{\partial f}{\partial x_n}$ is a regular sequence in R

$\Rightarrow R/df$ is finite dimensional.

A matrix factorization of f is a $\mathbb{Z}/2$ graded free R-module $M = M^0 \oplus M^1$ &

$$\begin{array}{ccccc} M^0 & \xrightarrow{D} & M^1 & \xrightarrow{D} & M^0 \\ \wr\!\| & & \wr\!\| & & \wr\!\| \\ R^k & \xrightarrow{B_1} & R^k & \xrightarrow{B_2} & R^k \end{array}$$

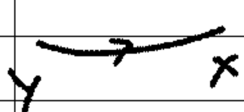
, D a module map & $D^2 = f$.

$\iff B_1, B_2$ square matrices with $B_1B_2 = B_2B_1 = f \cdot Id$

Later will need R local, $R = \mathbb{Q}[[x_1, \ldots x_n]]$

or $R = \mathbb{Q}[x_1 \ldots x_n]$ where $\deg x_i = 2$

& $\deg f = 2(n+1)$



let $f = x^{n+1} - y^{n+1} \in R = \mathbb{Q}[x,y]$

factorize as $R \xrightarrow{\pi_{xy}} R \xrightarrow{x-y} R$

$$\pi_{xy} = \frac{x^{n+1} - y^{n+1}}{x-y}$$

Close up arc $\iff$ set $x = y$

$\bigcirc \rightsquigarrow \mathbb{Q}[x] \xrightarrow{(n+1)x^n} \mathbb{Q}[x] \xrightarrow{0} \mathbb{Q}[x]$

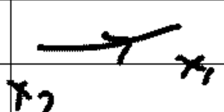
$\iff 0 \qquad \mathbb{Q}[x]/x^n$

Matrix factorizations localize the link homology:

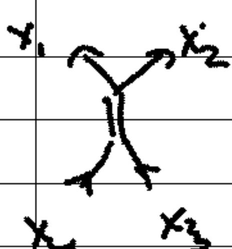
given $(M_i, D_i^2 = f_i)$ with $\sum f_i = 0$

$\Rightarrow \bigotimes_{i=1}^{n} M_i$ has $D^2 = 0$, ie honest complex

To a complicated graph in the plane we cut it into simple pieces & assign each a potential so that sum is zero



$$f = x_1^{n+1} - x_2^{n+1}$$



$$f = x_1^{n+1} + x_2^{n+1} - x_3^{n+1} - x_4^{n+1}$$ for some u_1, u_2

$$= (x_1 + x_2 - x_3 - x_4)u_1 + (x_1 x_2 - x_3 x_4)u_2$$

$[$like $x^{n+1} + y^{n+1} = g(x+y, xy)]$

$$f = g(x_1 + x_2, x_1 x_2) - g(x_3 + x_4, x_3 x_4)$$
$$- g(x_3 + x_4, x_1 x_2) + g(x_3 + x_4, x_1 x_2) \quad \cdots$$

...... analog of Koszul complex!

$$f = \sum a_i b_i \iff \bigotimes_{i=1}^{k} R \xrightarrow{b_i} R \xrightarrow{a_i} R = M(b,a)$$

while Koszul complex for $a_1, \dots, a_k$ is

$$\bigotimes_{i=1}^{k} \quad 0 \to R \xrightarrow{a_i} R \to 0$$

If $\sum a_i b_i = 0$ we'll say the pair (b,a) is <u>homologically regular</u> if $H^1(M(b,a)) = 0$

So above assignment in this notation is

$$e \longmapsto M_e = M(\underbrace{b}_{\parallel \; u_1, u_2}, \underbrace{a}_{= x_1+x_2-x_3-x_4,\; x_1x_2 - x_3x_4})$$

Thus to a graph Γ, we cut it into double edges & arcs $\Longrightarrow$

$$M_\Gamma = \bigotimes_{\text{double edges}} M_e \otimes \Big(\bigotimes_{\text{arcs}} M_{\frown} \Big)$$

2-periodic complex

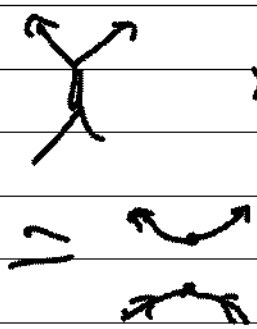
$H(M_\Gamma)$ is either $H^0(M_\Gamma)$ <u>or</u> $H^1(M_\Gamma)$

whose parity $\Longleftrightarrow$ # circles (modulo 2) in Γ.

$x_i \longleftrightarrow$ deg 2 generators in $H^*(\mathbb{CP}^{n-1})$

<u>Theorem</u> gr dim $H(\Gamma) = P_n(\Gamma)$

eg $n=2$



$x_1^3 + x_2^3 - x_3^3 - x_4^3$

$x_1^3 + x_2^3 = (x_1+x_2)(\dots)$

To graph $\rightsquigarrow$ matrix factorizations

tangle $\rightsquigarrow$ complex of matrix factorizations

link $\rightsquigarrow$ object of derived category of the triangulated category of matrix factorizations