M. Khovanov - Matrix factorizations etc IV

Note Title 1/29/2008

(w/ Lev Rozansky)

$R = k[[x_1 \ldots x_m]]$ or $k[x_1 \ldots x_m]$

$\ni$

f s.t. $R_f = R/(\frac{\partial f}{\partial x_i})$ finite dimensional
(i.e. regular sequence $\frac{\partial f}{\partial x_i}$)

MF_f category of matrix factorizations
with potential f.
$Hom_{MF}(M,N)$ is an R-module

HMF_f - homotopy category of matrix factorizations,
triangulated category,
& $Hom_{HMF}(M,N)$ is an R_f-module, since

since $D^2 = f \Rightarrow \frac{\partial D}{\partial x} D + D \frac{\partial D}{\partial x} = \frac{\partial f}{\partial x} Id$

so $\frac{\partial f}{\partial x}$ is homotopic to the identity.

<u>Theorem</u> (R-O Buchweitz) HMF is a
Calabi-Yau category:
$Hom_{HMF}(M,N) \times Hom(N,M) \to k$
natural nondegenerate pairing if m is even

If an odd natural pairing $\mathrm{Hom}(M,N)\times\mathrm{Hom}(N,M\langle 1\rangle)\to k$

Also $K(HMF)$ is (believed to be) torsion???

Can mod out a factorization by the maximal ideal:

$$M^0/\mathfrak{m}M^0 \xrightarrow{D} M^1/\mathfrak{m}M^1 \xrightarrow{D} M^0/\mathfrak{m}M^0$$

$H(M) :=$ homology of this complex.

A map $\alpha : M \to N$ is an isomorphism iff $H(\alpha): H(M)\to H(N)$ is an isomorphism

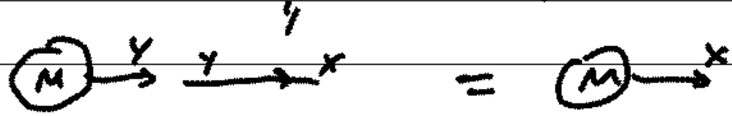
... so analog of homotopy & of derived category

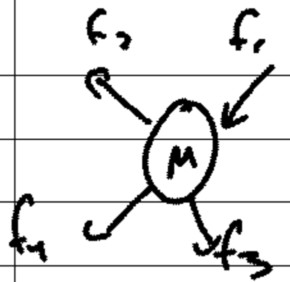
Category is Krull Schmidt: every object is $\oplus$ indecomp. in unique way, & have canonical minimal representatives.

Let L_{xy} be the factorization of $f(x) = x^n - y^n$

$$R \xrightarrow{\frac{x^{n+1}-y^{n+1}}{x-y}} R \xrightarrow{x-y} R$$

Then $M_y \otimes_y L_{xy} \simeq M_x$ 'identity functor'



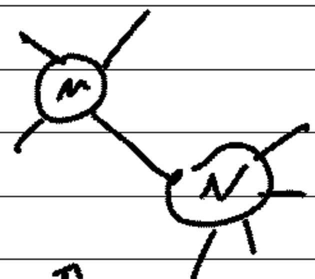


picture for any factorization of
$f = \sum f_i$

.... can glue to

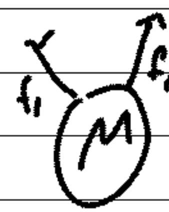
over common polynomial

$\Rightarrow$



$M \underset{x_1 \ldots x_n}{\otimes} N$ matrix factorization of resulting sum polynomial.

or



$\simeq$

$M/(x_i - \hat{x}_i)$

More generally if have $f(x_1 \ldots x_m)$,
write $f(x_1 \ldots x_m) - f(y_1 \ldots y_m) = \sum (x_i - y_i) u_i$

$\Rightarrow L_{xy} = \bigotimes_{i=1}^{m} \left(R \xrightarrow{u_i} R \xrightarrow{x_i - y_i} R\right)$

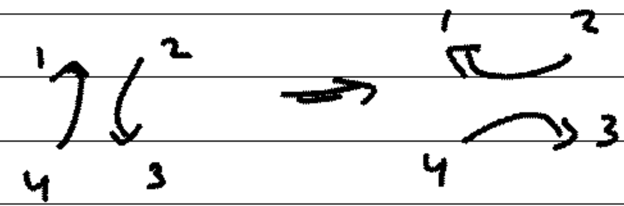
represents the identity (analog of diagonal):

$M \underset{y}{\otimes} L_{xy} \simeq M$

To $f = x^{n+1}$ assign extended 2d TQFT

$\bullet \; \bullet \; \bullet \; \bullet \longmapsto \sum \pm x_i^{n+1}$

$\bigcirc \longmapsto R_f = k[x]/(x^n$

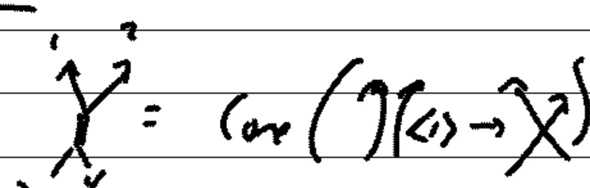


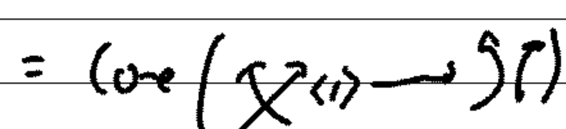
assign

$$\begin{matrix} R \xrightarrow{\pi_{14}} R \xrightarrow{x_1 - x_4} R \\ \otimes \\ R \xrightarrow{\pi_{23}} R \xrightarrow{x_3 - x_2} R \end{matrix} \Rightarrow \begin{matrix} R \xrightarrow{\pi_{12}} R \xrightarrow{x_1 - x_2} R \\ \otimes \\ R \xrightarrow{\pi_{34}} R \xrightarrow{x_3 - x_4} R \end{matrix} \quad \langle 1 \rangle$$

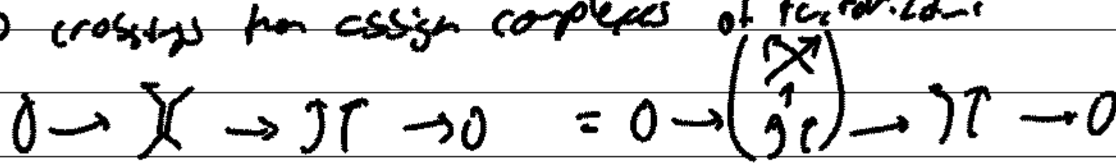
$\cdots$

Link homology: we assigned



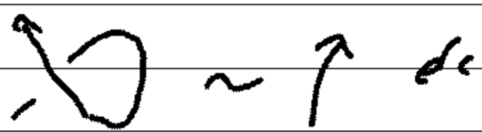


To crossings then assign complexes of factorizations



⇝ look in category of complexes / chain homotopies in the category HMF_f

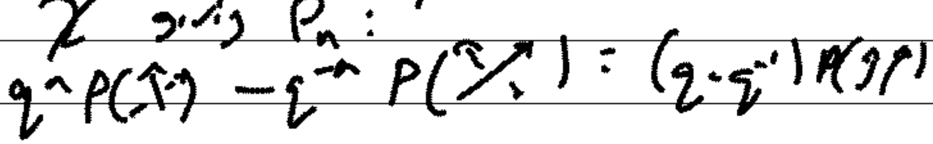
Find in here Reidemeister moves are satisfied:

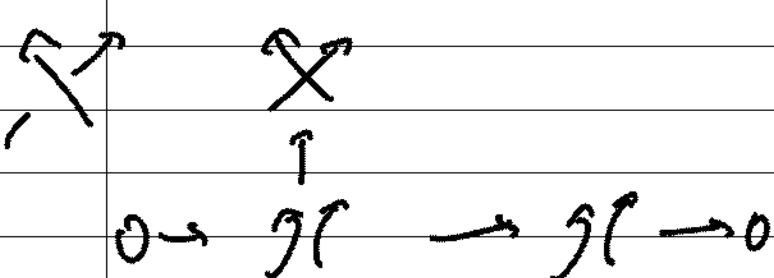


⇒ assign complexes $F(T)$ to tangles with orientation.

[Need char $k \nmid n+1$:
want x^{n+1} to have nonzero derivative ...]

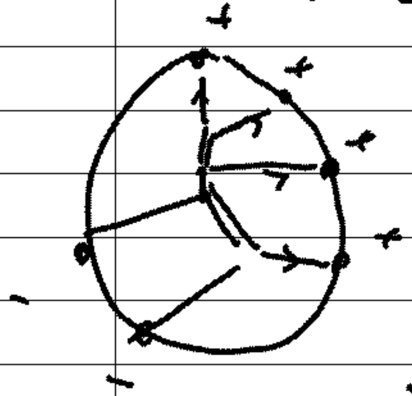
⇒ bigraded homology theory for links with χ giving P_n:





homological perturbation of permutation [crossing] which miraculously satisfies Reidemeister moves.

$f = \sum I x_i^{n+1}$



To any graph Γ with given boundary can assign an invariant tensor

$$m(\Gamma) \in \mathrm{Inv}(V \otimes V \dots \otimes V^* \otimes V^* \dots)$$

After categorification get $M(\Gamma) \in HMF_f$

but this has a bad K-group....

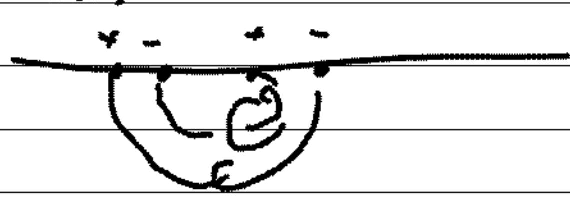
but can build abelian category by summing over all such graphs,

$$\& \quad A = \bigoplus_{i,j \in I} \mathrm{Hom}(M(\Gamma_i), M(\Gamma_j))$$

with I some (conjecturally finite) collection of Γ with $m(\Gamma)$ spanning space of invariants

$\Rightarrow$ conjecturally f.g. proj A-modules $\subset HMF_f$

Tangles: should categorify spaces of invariants of tensor products: $T \mapsto F(T) \in \mathcal{C}$ with $K(\mathcal{C})$ = invariants



Conjecture Direct summands of $M(\Gamma)$ ($\forall$ such Γ) give Lusztig dual canonical basis in invariant space $\mathrm{Inv}(V \otimes \dots V^* \otimes \dots)$

$\rightsquigarrow$ replace $C(HMF_f)$ with Complexes of A-modules, much nicer category.

But eg A-mod won't include things like