Jacob Lurie - Intro to Derived Algebraic Geometry

Note Title 3/27/2008

Motivation: Virtual Fundamental Classes

Suppose $C, C' \subset \mathbb{CP}^2$ curves of degrees m, n.

Bezout: If C, C' meet transversally $\Rightarrow$

$\#(C \cap C') = nm$ ....

equality in $H^4(\mathbb{P}^2, \mathbb{Z}) \simeq \mathbb{Z}$

$[C], [C'] \in H^2(\mathbb{P}^2, \mathbb{Z}) \simeq \mathbb{Z}$ (degree)

$$\underset{\underset{[C \cap C']}{\shortparallel}}{\#(C \cap C')} = nm = [C] \cup [C'] \in H^4(\mathbb{P}^2, \mathbb{Z})$$

: cup of fundamental classes is class of intersection.

If $C \not\pitchfork C'$, then $\# C \cap C' < nm$.

... need to take into account multiplicities:

$$\sum_{p \in C \cap C'} IM_p(C, C') = nm$$

where intersection multiplicity is

$$IM_p(C, C') := \dim\left(\mathcal{O}_{C,p} \underset{\mathcal{O}_{\mathbb{P}^2,p}}{\otimes} \mathcal{O}_{C',p} \right)$$

dim of tensor of local rings.

$$[C \cap C'] = [C] \smile [C'] \text{ as } \underline{\text{schemes}}:$$

count points with multiplicity as dims of local rings: interpret LHS correctly.

Suppose $C, C' \subset \mathbb{P}^d$ where $\dim C + \dim C' = d$, of degrees n, m.

If C, C' meet properly ($\dim C \cap C' = 0$)

$$\Rightarrow \quad nm = \sum_{p \in C \cap C'} IM_p(C, C')$$

Serre's multiplicity formula:

$$IM_p(C, C') = \sum (-1)^i \dim \mathrm{Tor}_i^{\mathcal{O}_{\mathbb{P}^d, p}} (\mathcal{O}_{C,p}, \mathcal{O}_{C',p})$$

— higher order corrections to multiplicity

Scheme theoretic intersection doesn't remember higher Tors: should take $C \cap C'$ in a more precise sense $\rightsquigarrow$ DAG:

recover $[C \cap C'] = [C] \smile [C']$

Go back to $\mathbb{P}^2$ & let's intersect lines:

$L, L' \subset \mathbb{P}^2$: always meet transversally unless they coincide: if $L = L'$

$L \cap L' = L$: wrong dimension!

so $[L \cap L']$ has wrong degree.

To get a good theory, want to work in a setting where $L \cap L \neq L$.

Look locally at $\mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x,y] \subset \mathbb{P}^2$.

If $L \neq L'$, then WLOG can take

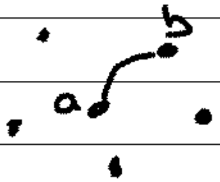
$L : (x=0) \qquad L' : (y=0)$

$L \cap L' = \operatorname{Spec} \mathbb{C}[x,y]/(x,y) \simeq \operatorname{Spec} \mathbb{C}$

$L = L'$: WLOG $L = L' : (x=0)$

$L \cap L' = \operatorname{Spec} \mathbb{C}[x,y]/(x,x) \simeq \operatorname{Spec} \mathbb{C}[y]$

Ignore ring structures: imposing equation on sets $\longleftrightarrow$ equivalence relations



Think of sets as discrete spaces
& impose equivalence relations
by adding paths

Can now set $a=b$ twice: 

no longer equivalent to a discrete space,
remembers two identifications.

A CW complex is a space built from $\emptyset$
by adding cells of various dimensions in sequence
... presenting sets by generators &
relations! 0-cells give initial set, add
1-cells to impose relations. On π_0 that's all
we care..

Algebraic topology $\Leftrightarrow$ more sophisticated notion
of generators & relations.

Idea: to get derived algebraic geometry,
work with "commutative rings in algebraic topology"
instead of usual comm. rings.

Most naive notion of comm. ring in topology:
topological comm. ring := comm. ring R
with a topology st $+, \cdot$ are continuous.
$\rightsquigarrow$ simplicial commutative rings

Words on topological comm. rings:
only interested in the topology as a mechanism
for talking about homotopy groups:
$f: A \to B$ is an equivalence if f induces
$$\pi_i A \xrightarrow{\sim} \pi_i B \quad \forall i \geq 0$$
(based at identity)

In this case A is "just as good as" B

If A is a top. comm. ring, $\pi_0 A$ is an
ordinary comm. ring — "underlying commutative ring"
$A \longrightarrow \pi_0 A$ map in homotopy category, collapsing
components
This is an equivalence iff higher $\pi_i A$ vanish.

$\pi_i A$ are $\pi_0 A$ modules, in fact $\pi_* A$ is a graded-commutative ring.

<u>Def</u> A derived scheme is a topological space X w/ a sheaf $\mathcal{O}_X$ of topological comm rings s.t.

$(X, \mathcal{O}_X)$ <u>is</u> locally equivalent to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$
A topological commutative ring.

"<u>Sheaf</u>": <u>don't</u> mean sheaf in <u>category</u> of topological comm rings – should think of them as an $(\infty,1)$ category:
given A, B there is a <u>space</u> $\operatorname{Map}(A,B)$

... as first approximation look at all maps $A \to B$ with compact-open topology
$\supset$ closed subset (for B Hausdorff) of <u>ring</u> maps.

This is correct if A is nice enough: <u>cofibrant</u>: given by cell attachments from the empty set in the world of comm. top. rings — otherwise need to resolve A.

$\rightsquigarrow$ meaning of sheaf must be adjusted to setting of $(\infty,1)$ categories

<u>Rmks</u> • $\mathrm{Map}_{ss}(X, \mathrm{Map}(A,B)) = \mathrm{Map}(A, B^X)$
for X a simplicial set.

• Cofibrant replacement: take geometric realization of singular cochains: both commute with finite products so compatible with ring structure

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Classical def of sheaf of sets:
$\mathcal{F}$ presheaf on X is a sheaf $\Rightarrow$
given $U, V \subset X$

$$\begin{tikzcd} \mathcal{F}(U \cup V) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r] & \mathcal{F}(U \cap V) \end{tikzcd}$$

is a pullback diagram

(+ other properties, also diagrammatic!)

In ∞ categorical case:

First, might modify notion of presheaf
- require compatible with composition
up to homotopy, coherently ..... can always
Strictify to be compatible on the nose.

More important: modify sheaf condition:

$$\begin{array}{ccc} F(U \cup V) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U \cap V) \end{array}$$

homotopy pullback:

$$Z \longrightarrow F(U \cup V) \rightrightarrows F(U \cap V)$$

get two homotopic maps to $F(U \cap V)$:
want universal property to be

$$Z \to F(U \cup V) \underset{\text{homotopy}}{\simeq} \left\{ \begin{array}{l} Z \to F(U) \\ Z \to F(V) \end{array} + \text{homotopy} \right\}$$

equivalence of induced $Z \rightrightarrows F(U \cap V)$

[Note : if $\mathcal{F}(U)$ discrete $\forall U$, this is the usual sheaf axiom]

<u>Def</u> Let A be a top. comm. ring

Spec A := Spec $\pi_0 A$ (Zariski spectrum)

$= \{ \mathfrak{p} \subseteq \pi_0 A \text{ prime} \} \supset U_f = \{ \mathfrak{p} \not\ni f \}$

basis of opens

$\mathcal{O}_{\mathrm{Spec}\,A}(U_f) = A[f^{-1}]$ localization!

<u>Def</u> A a top. comm. ring & $f \in \pi_0 A$

$A[f^{-1}]$ is a top. comm ring with a map

$A \longrightarrow A[f^{-1}]$, universal among maps

inverting f : $\forall B$ $\mathrm{Map}(A[f^{-1}], B) \to \mathrm{Map}(A, B)$

is a homotopy equivalence onto connected components containing $\varphi : A \to B$ where

$f \mapsto$ invertible in $\pi_0 B$.

$\Longleftrightarrow$ Construct by adjoining inverse x to f

$A[x]$ & then kill it in more sophisticated sense: join xf to 1 by a path

... construction depends on choice of lift of f to A — but canonically indep of choice thanks to universal mapping property
$\Leftrightarrow$ invert every element of the connected component simultaneously.

$\Leftrightarrow$ there's an isomorphism $(\pi_* A)[f^{-1}] \xrightarrow{\sim} \pi_*(A[f^{-1}])$

(localization classically is exact $\rightsquigarrow$ if A discrete this is usual discrete localization)

Observation: This presheaf extends to a sheaf on Spec A.
(localization enjoys descent property)

Example Any ordinary scheme is a derived scheme.

[having characteristic p is an additional datum in this context!]

Adjoint construction: $(X, \mathcal{O}_X)$ derived scheme $\Rightarrow$ $(X, \pi_0 \mathcal{O}_X)$ scheme [need to sheafify π_0]

.. "underlying ordinary scheme":
$(X, \mathcal{O}_X)$ gives an ordinary scheme $(X, \pi_0 \mathcal{O}_X)$
+ higher homotopy groups $\pi_i \mathcal{O}_X$ which are quasicoherent sheaves on $(X, \pi_0 \mathcal{O}_X)$
+ gluing data...

<u>Analogy</u> Derived schemes : schemes :: schemes : reduced schemes

$$(X, \mathcal{O}_X) \supset (X, \pi_0 \mathcal{O}_X) \qquad X \supset X^{red}$$

$$A \longrightarrow \pi_0 A$$

"largest ordinary subscheme in $(X, \mathcal{O}_X)$"

In world of derived schemes have notion of intersection: fiber product

$$\begin{tikzcd} C \cap C' \arrow[r] \arrow[d] & C \arrow[d, hook] \\ C' \arrow[r, hook] & \mathbb{P}^d \end{tikzcd}$$

fiber product of derived schemes

+ formula $[C \cap C'] = [C] \cup [C']$
always true provided $[C \cap C']$ defined correctly

(virtual fundamental class ... need cotangent complex to have Tor amplitude 1 ... eg intersection of quasismooth inside smooth ... defined integrally)

X scheme/R : view X as a functor

$\underline{X}$: comm R-alg $\longrightarrow$ Sets

$A \longmapsto \mathrm{Hom}_{R\text{-schemes}}(\operatorname{Spec} A, X)$

X is a group scheme over R

$\Longleftrightarrow$ $\underline{X}$ factors through Gps.

R a top. comm ring, X a derived scheme

$X \longrightarrow \operatorname{Spec} R$

$\longrightarrow$ functor $\left\{\begin{matrix}\text{top.}\\ R\text{-algebras}\end{matrix}\right\} \longrightarrow \{\text{Spaces}\}$

$A \longmapsto \underset{R\text{-sch}}{\mathrm{Map}}(\operatorname{Spec} A, X)$.

Group scheme: $\underline{X}$ factors through various notions of groups ... eg topological groups, top. abelian groups, ∞ loop spaces etc.

Moduli of vector bundles

Given a top. comm. ring $A \Rightarrow$ "category" of topological A-modules ... more precisely eg a simplicial model category $\mathbb{A}$ with objects = simplicial A-modules.

Inside $\mathbb{A}$, can consider fibrant-cofibrant objects M s.t. $\pi_* M$ is a projective module over $\pi_* A$ of rank n.

$\rightsquigarrow$ get a subcategory $\mathcal{C}_A$

Functor $A \longmapsto B\mathcal{C}_A$: not strictly functorial: can tensor A-modules up to B, ...

need to work a little to make strictly functorial

$\rightarrow$ functor representing rank n vector bundles.

Smoothness Let $R \to R'$ be a map of top. comm. rings

Def F is flat if

- $\pi_0 R \longrightarrow \pi_0 R'$ is flat
- $\pi_0 R' \otimes_{\pi_0 R} \pi_i R \xrightarrow{\sim} \pi_i R'$

$\iff$ tensoring is t-exact
$\iff$ filtered colimit of free R-modules
(on modules rather than algebras)

<u>Def</u> Let P be a property of maps of comm. rings which implies flatness (eg étale, smooth, ...) Then we say $f: R \to R'$ has property P if
- f is flat
- $\pi_0 R \to \pi_0 R'$ has property P.

X variety $\rightsquigarrow$ $\exists$ derived moduli stack $\mathcal{M}$ of "vector bundles on X"
- classically, if $\dim X = 1$, $\mathcal{M}$ is a smooth Artin stack
- if $\dim X > 1$ $\mathcal{M}$ is "derived" but its cotangent complex has Tor-amplitude bounded by $n-1$.

(Tor amplitude $= 1$: "quasismooth")

Other kinds of derived schemes

good for topology

Topological comm rings
$\simeq$
simplicial comm rings $\xrightarrow{\text{equiv in char. 0}}$ comm. dg algebras (cohomology only in ≤ 0 degrees) $\xrightarrow{\text{equiv in char 0}}$ E_∞ ring spectrum (connective)

good for algebra

Connective E_∞ ring spectrum: topological space with ring structure comm & assoc up to coherent homotopy

$\longleftrightarrow$ cohomology theories with good multiplicative properties – eg (connective) complex K-theory

Pitfalls for DAG over E_∞ ring spectra:

There are two candidates for $\mathbb{A}^1$.

Let's work over a field k of characteristic p.

Candidate #1: Usual $\mathbb{A}^1 = \operatorname{Spec} k[x]$ ($k[x]$ as discrete ring) --- this is flat $/k$.

Second candidate: Spec $k\{x\}$: free E_∞ algebra / k on one generator.

– don't coincide in char. p :

In general given V vector space / k

$$\mathrm{Sym}^* V = \bigoplus_{n\geq 0} \mathrm{Sym}^n V = \bigoplus_{n\geq 0} \left(V^{\otimes n}\right)_{\Sigma_n} \quad \text{coinvariants}$$

free comm algebra in usual world.

E_∞ world: use homotopy coinvariants – see higher homology of Σ_n appearing.

$$\pi_n \, k\{x\} = \bigoplus_m H_n(\Sigma_m, k)$$

If char $k \neq 0$ this doesn't vanish for $n > 0$
→ different version of polynomial ring, so not flat, but is smooth in some sense (infinitesimal lifting)

– smooth things are typically not flat.