Jacob Lurie - Topological Field Theory in Low Dimensions

Note Title 3/3/2008

(joint with M. Hopkins - inspired by K. Costello)

Manifolds are always (for us) smooth, compact & oriented (often w/ boundary)

nCob : category with objects $(n-1)$ manifolds (closed) & morphisms: bordisms of $(n-1)$ manifold up to diffeomorphisms (rel boundary)

- this has a tensor product (symmetric monoidal) given by disjoint union $\underline{\amalg}$

Def. (Atiyah) An (n-dim) TQFT is a $\otimes$-functor
$F : n\mathrm{Cob} \longrightarrow \mathbb{C}$ vector spaces

ie $F(M^{n-1})$: vector space
& $F(M \amalg N) \xrightarrow{\sim} F(M) \otimes F(N)$

$F(\emptyset) = \mathbb{C}$ unit property

Example $n=2$: every closed 1-manifold is a $\amalg$ of circles

$F($$) = A$ vector space

Functoriality $\Rightarrow$ $F($$) : A \otimes A \xrightarrow{m} A$

m is commutative & associative

Unit: $F($$) : \mathbb{C} \longrightarrow A$

$\Rightarrow$ element $1 \in A$

Trace: $F($$) : A \xrightarrow{tr} \mathbb{C}$

nondegenerate:

$$A \otimes A \xrightarrow{m} A \xrightarrow{tr} \mathbb{C}$$

is a perfect pairing

$\Longrightarrow$ A is a commutative Frobenius algebra

Folk theorem: there's a converse to this:

Given A comm. Frobenius algebra

$\rightsquigarrow$ construct 2d TFT F.

Another POV: fundamentally F (n-dim TQFT) takes values on closed n-manifolds M

$$F(M): \mathbb{C} = F(\emptyset) \longrightarrow \mathbb{C}$$

given by multiplication by a number, which is a diffeomorphism invariant of M.

Extra structure: set of rules to compute these numbers by cut & paste

In higher dimensions these rules are limited in scope ... would like to chop very finely, eg by a triangulation — pieces with corners — so as to reduce the problem to very local data.

<u>Def.</u> (sketch) An extended TQFT in dimension n is a rule

closed n-manifold $\rightsquigarrow$ complex number

closed $(n-1)$ manifold $\rightsquigarrow$ complex vector space

bordism of " " $\rightsquigarrow$ linear maps

[consistency: closed n-manifolds $\leftrightarrow$ bordisms of empty $n-1$ manifold] keep going!

closed $(n-2)$-manifolds $\rightsquigarrow$ $\mathbb{C}$-linear category

bordisms of $(n-2)$ manifolds $\rightsquigarrow$ $\mathbb{C}$-linear functors

[consistency: closed $n-1$ manifold = bordism of empty $n-2$ manifolds $\Rightarrow$ invariant is multiplication by a vector space]

& so on ..... for us we'll only take $n \leq 2$ so won't need to keep going

Gluing rules — summarized as a functor between "n-categories"

Perspective (Baez-Dolan): [paraphrased]

<u>Cobordism hypothesis</u> Extended TQFTs are "easy to describe/build"

Philosophy behind it: n-manifolds are very simple locally: simplices or discs or points...

Functor between n-categories $\longleftrightarrow$ set of rules determining $F(M)$ in terms of elemental building blocks, like points.

Another reason to introduce higher categories into TQFTs (with $n=2$)

[Non]example [$n=2$] : String Topology
Let M be a manifold. Then there (almost) exists a 2d TQFT F s.t.

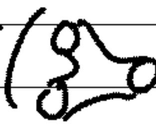
$$F(\bigcirc) = H_*(LM; \mathbb{C})$$

$LM = \mathrm{Map}(S^1, M)$ loop space

.... "noncompact" field theory (in G. Segal terminology): given a bordism

$$I \xrightarrow{\Sigma} J \quad \text{get} \quad F(I) \xrightarrow{F(\Sigma)} F(J)$$

provided every component of Σ has nonempty intersection with J

eg $F($  $)$ gives Chas-Sullivan product on $H_*(LM)$

but $F(\bigcirc)$ not allowed.

no trace since $H_*(LM)$ typically infinite dimensional.

Example Let X be a Calabi-Yau variety/$\mathbb{C}$
Then $\exists$ 2d TQFT with
$F(\ominus) = HH_*(X)$ [topological B-model]

Example (Freed-Hopkins-Teleman on twisted K-theory)
G compact simple Lie group, simply connected
$l \in H^4(BG;\mathbb{Z}) \simeq \mathbb{Z}$ positive level

$\exists$ good theory of projective level l representations of LG --- finitely many isom classes of representations
$\Rightarrow$ representation ring A (= Verlinde algebra)

FHT: $A \simeq K_G^{\tau(l)}(G)$ twisted equivariant K-group.
Verlinde product etc $\longleftrightarrow$ natural operations in twisted K-theory

$\exists$ 2d TQFT F with $F(\bigcirc) \simeq A$.
(fin gen abelian group rather than $\mathbb{C}$ vector space)

All three examples have an algebraic topology flavor: first two in fact come from the homology of a chain complex.
Can try to lift the operations to chain level!

<u>Def</u> A (chain-complex valued) TQFT F assigns
$(n-1)$-manifold $M \rightsquigarrow$ chain complex $F(M)$
bordism $B: M \to N \rightsquigarrow$ map of chain complexes
$\vdots$

More systematic description:
Let $\mathrm{Bord}(M,N)$ = classifying space for bordisms $M \rightsquigarrow N$:
topological space with fiber bundle over it of bordisms $M \rightsquigarrow N$, & <u>universal</u> for this property.

We get a map $C_*(\mathrm{Bord}(M,N)) \to \mathrm{Hom}(F(M),F(N))$
map of chain complexes

e.g path between two bordisms gives a <u>chain homotopy</u> between the corresponding maps of chain complexes

This gives rise to interesting structure:

$$C_*(\mathrm{Bord}(M,N)) \otimes F(M) \longrightarrow F(N)$$

$$\Rightarrow \quad H_*(\mathrm{Bord}) \otimes H_* F(M) \longrightarrow H_* F(N)$$

H_0 gives just the maps corresponding to components, ie bordism / isotopy — which is what we had before.

But higher homologies give interesting new operations!

e.g $n=2$ these classifying spaces are roughly moduli spaces of curves $\longrightarrow$

get interesting reps of the homology of mapping class groups.

<u>Reformulation</u> We have a map

$$\mathrm{Bord}(M,N) \longrightarrow \mathrm{Map}(F(M), F(N))$$

map of topological spaces — manifesting analogy between topology & homological algebra, ..

e.g. $\pi_i \mathrm{Map}(F(M), F(N))$

$\cong$ chain homotopy classes of maps $F(M)[i] \longrightarrow F(N)$.

Can think of F as a functor between categories but now they are <u>topological</u> categories: Homs are <u>spaces</u>.

---- model for higher categories in which all higher morphisms are invertible

Main definition (Sketch)

nBord : "higher category"

Objects : 0-manifolds

morphisms : bordisms of 0-manifolds

2-morphisms : bordisms of bordisms

. . . .

n-morphisms : (n)-manifolds w/corners

$(n+1)$-morphisms : diffeomorphisms

$(n+2)$-morphisms : isotopies of diffeos . .

(ie starting from $n+1$ get a <u>space</u> of diffeomorphisms of manifolds ...)

— more concretely next time! $(n=1,2)$.

$\sqcup$ makes n-Bord into a symmetric monoidal (higher) category

<u>Problem</u> Describe $\otimes$-functors

$n\text{Bord} \longrightarrow \mathcal{C}$ where $\mathcal{C}$ is again a higher category with $\otimes$

Lecture 1 : ✓

Lecture 2 : Start with sketch of higher category theory, discuss the problem & the solution for $n=1$

Lecture 3: Discuss solution for $n=2$, make contact with work of Costello & Galatius - Madsen - Tillman - Weiss

Lecture 4: Applications (i.e. produce string topology...) & extensions to higher dimensions