Jacob Lurie - TFT in low dimensions 2

Note Title 3/5/2008

Overview of higher categories

Recall: A category $\mathcal{C}$ consists of
- some objects X, Y, Z
- For $X, Y \in \mathcal{C}$ a set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$
- composition $\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z)$
- conditions: units, associative, ..

If S is any category with products, can consider S-enriched categories

An S-enriched category $\mathcal{C}$
- some objects X, Y, Z
- For $X, Y \in \mathcal{C}$ an object $\mathrm{Hom}_{\mathcal{C}}(X,Y) \in S$
- composition $\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z)$

  morphisms in S
- conditions: units, associative, ..

Naive definition of higher categories:
For $n > 0$ a strict n-category is a category enriched over strict $(n-1)$ categories.

($n=0$: a strict n-category is a set)

$n=1 \Rightarrow$ category
$n=2 \Rightarrow$ for every pair of objects get a Hom category whose objects we call 1-morphisms & Homs we call 2-morphisms.

Why is this naive?
associative law

$$\begin{array}{ccc} \mathrm{Hom}(W,X) \times \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \\ \downarrow \quad \downarrow \\ \mathrm{Hom}(W,Z) \end{array}$$

two maps
$(()\,)$ or $(\,())$

But these Homs are categories if $n=2$: should never say two functors are <u>equal</u> - should specify an isomorphism
$\Rightarrow$ associativity up to isomorphism
+ coherence conditions

Example (of a less naive notion)

Let X be a topological space

Fundamental groupoid $\Pi_{\leq 1} X$:

category with objects : points in X

morphisms : paths in X / homotopy

– encodes $\pi_0 X$, $\pi_1 X$.

$\Pi_{\leq 2} X$ fundamental 2-groupoid :

2-category, objects = points of X

morphisms = paths in X

2-morphisms = homotopies of paths / homotopy

Difficulty : A path from x to y in X is a map $p : [0,1] \longrightarrow X$

$p(0) = x \quad p(1) = y$

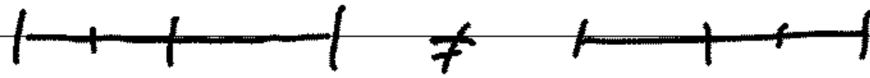
Composition $x \xrightarrow{p} y \xrightarrow{q} z$

$q \circ p : [0,1] \longrightarrow X$:

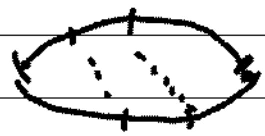
$[0 \underbrace{\cdots\cdots}_{p} \mid \underbrace{\cdots\cdots}_{q} 1]$

twice as fast

$\Rightarrow$ associativity fails



— need isomorphism for associativity



(can avoid by defining paths as not maps from fixed interval but from any interval)

More such invariants $\Pi_{\leq 3}X$, $\Pi_{\leq 4}X$ ...

e.g. $\Pi_{\leq 3}X$ can <u>not</u> be defined as a strict 3-category in general! so need more sophisticated notion.

There is a notion of n-category for each $n \geq 0$. Very natural to parametrize by two indices

<u>Def</u> For $m \geq n$, an (m,n)-category is an m-category where all k-morphisms are invertible for $k > n$.

Growth of complexity dramatic in n, but not in m!
Invertible higher morphisms not hard to think about.

Ex. $n\mathrm{Bord}$: ob = 0-manifolds
1-mor = bordisms
2-mor = " between bordisms
- - -
cut off at level n: everything above degree n is invertible
($n+1$ morphisms are diffeos)

$\Rightarrow$ an (∞, n) category.

Ex. X topological space,
$\pi_{\leq n} X$ is an $(n,0)$-category or n-groupoid.

<u>Thesis</u> n-groupoids = $(n,0)$-categories / equivalence

$\simeq$

topological spaces with no homotopy groups above n / homotopy equivalence

$(\infty,0)$ categories $\longleftrightarrow$ topological spaces / homotopy equivalence

... should become a theorem every time we give a definition of the LHS.

$\rightsquigarrow$ <u>Definition</u> An $(\infty,0)$ category is a topological space. ($\leftrightarrow$ simplicial set $\leftrightarrow$ Kan complex)

<u>Definition</u> (correct but inconvenient)
An $(\infty,1)$ category is a category enriched over topological spaces
... ie Homs are <u>spaces</u>, strictly associative composition.

Correct: can always strictify a looser definition into an example of this
inconvenient: will have to always fiddle & strictify to get things to fit in this framework.

<u>Example</u> 1Bord (2nd attempt at a definition)

Objects: 0-manifolds [everything compact, oriented]

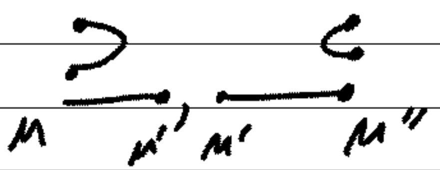
Morphisms: $\mathrm{Hom}_{|\mathrm{Bord}|}(M, M')$ =
classifying space for bordisms from M to M'.

Composition: $\mathrm{Hom}_{|\mathrm{Bord}|}(M,M') \times \mathrm{Hom}_{|\mathrm{Bord}|}(M',M'')$

$$\downarrow$$

$\mathrm{Hom}_{|\mathrm{Bord}|}(M,M'')$

... over this product space have two universal bundles,



so can glue ⟹ get a classifying map to $\mathrm{Hom}(M, M'')$

Problem w/ associativity: everything defined only up to homotopy
- possible to "straighten things out" & get a strict associativity

Another definition for $(\infty,1)$ categories
(complete Segal spaces)

Suppose $\mathcal{C}$ is an $(\infty,1)$ category

Step 1. $\mathcal{C}$ has an underlying $(\infty,0)$ category:
throw away all noninvertible morphisms,
get a <u>space</u> X_0.

Step 2 Remember noninvertible morphisms:
morphisms are functors $\mathrm{Fun}(\{0,1\}, \mathcal{C})$
($\{0,1\}$ two objects & 1 morphisms)
should form an $(\infty,1)$ category,
now throw away noninvertible morphisms
$\Rightarrow$ get a space X_1 classifying
morphisms in $\mathcal{C}$

Remember chains of morphisms:
Consider $\mathrm{Fun}(\{0,\dots n\}, \mathcal{C})$
chains of length n & throw away
noninvertible morphisms $\longrightarrow$ space X_n.

($\{0, \dots n\}$: linearly ordered set as category)

Any monotone map
$\{0, \dots m\} \to \{0, \dots n\}$ induces
a composition $X_n \longrightarrow X_m$

ie we have a simplicial space $\{X_n\}_{n \geq 0}$

Special features of this simplicial space

$$\begin{tikzcd}
\text{comm. triangles} \arrow[r, Rightarrow] & X_2 \arrow[r, "\text{extract first arrow}"] \arrow[d, "\text{extract second arrow}"'] & X_1 \arrow[d, "\text{target}"] & \text{morphisms} \\
& X_1 \arrow[r, "\text{source}"'] & X_0 & \\
& \text{morphisms} & & \text{objects}
\end{tikzcd}$$

$\leadsto$ map $X_2 \longrightarrow X_1 \underset{X_0}{\times} X_1$

In an ordinary category this would be an isomorphism ... we'll demand this is a homotopy pullback square

$\rightsquigarrow$ composition of morphisms, up to contractible ambiguity

$\Rightarrow$ can extract the long arrow $X_2 \longrightarrow X_1$ giving composition.

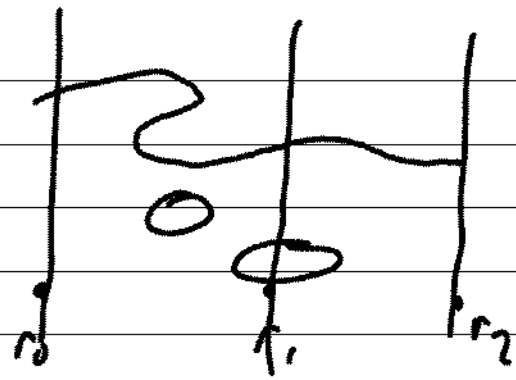
Def A complete Segal space is a simplicial space $\{X_n\}$ satisfying some conditions (homotopy pullbacks for diagrams as above)

In fact topological categories $\longleftrightarrow$ complete Segal spaces

Example 1Bord as a complete Segal space: ie define spaces X_n $n \geq 0$, "spaces of n-tuples of composable bordisms"

$X_n = \{ (r_0 \leq \ldots \leq r_n)$ real numbers, $M \subset [r_0, r_n] \times \mathbb{R}^\infty$ properly embedded 1-manifold with boundary & M meets $\{r_i\} \times \mathbb{R}^\infty$ transversally $\}$

e.g. X_2



looks like morphisms $\bullet \longrightarrow \begin{matrix}\bullet\\ \vdots\end{matrix}$

& $\begin{matrix}\bullet\\ \bullet\\ \vdots\end{matrix} \longrightarrow \bullet$ and can see composition

1Bord (in this incarnation) + higher dimensional relatives (for manifolds of any dimension) was studied by Galatius, Madsen, Tillmann & Weiss for a slightly different purpose:

they computed the geometric realization $|X_\bullet|$ in any dimension... in dim 1

$$|X_\bullet| \simeq \varinjlim \Omega^n S^n \quad \text{stable sphere}$$

– satisfies a very simple universal property.

Problem Describe tensor functors

$\mathrm{Fun}^{\otimes}(1\mathrm{Bord}, \mathcal{C})$ $\quad$ $\mathcal{C}$ some tensor $(\infty,1)$ cat.
(symmetric monoidal)

Suppose $F : 1\mathrm{Bord} \to \mathcal{C}$.

Let's evaluate: $F(\emptyset)$ must be the unit $\underline{1}_{\mathcal{C}} \in \mathcal{C}$

$F(\overset{+}{\bullet}) = C \in \mathcal{C}$

$F(\overset{-}{\bullet}) = D \in \mathcal{C}$

But in fact D is determined by C by duality:

$F(\supset) : C \otimes D \to 1_{\mathcal{C}}$

$F(\subset) : 1_{\mathcal{C}} \to C \otimes D$

these exhibit C & D as dual so can recover D as dual of C $\rightsquigarrow$ just need to know that C is <u>dualizable</u>.

<u>Theorem</u> $Fun^{\otimes}(1Bord, \mathcal{C}) \simeq$ dualizable objects of $\mathcal{C}$
[with invertible morphisms]
equivalence of $(\infty,1)$ category.

Given C want to build F:

$F(\overset{+}{\bullet}\ \overset{+}{\bullet}\ \underset{-}{\bullet}) $ must be $C \otimes C \otimes C^{\vee}$ by monoidal property

Need to say what F does on connected bordisms:

$F(\longrightarrow) = id_C \qquad F(\longleftarrow) = id_{C^{\vee}}$

$F(\supset) = ev : C \otimes C^{\vee} \longrightarrow 1$

$F(\subset) = coev : 1 \longrightarrow C \otimes C^{\vee}$

$$F(\bigcirc) = 1 \xrightarrow{coev} C \otimes C^{\vee} \xrightarrow{ev} 1$$

$$= \text{"}\dim C\text{"} \in Hom_{\mathcal{C}}(1,1)$$

... eg if $\mathcal{C}$ was Vect
this is trace of the identity matrix,
ie the dimension

Why is this <u>non</u> obvious: 1Bord, $\mathcal{C}$ are not ordinary categories!

Consider eg $\mathrm{Hom}_{1\mathrm{Bord}}(1,1)$

= classifying space for all closed 1-manifolds

$\cup$

[connected 1-manifolds] classifying space for oriented circle bundles, $\simeq \mathbb{C}P^\infty$.

$$\Rightarrow \quad \begin{array}{ccc} \mathbb{C}P^\infty \subset \mathrm{Hom}_{1\mathrm{Bord}}(1,1) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(1,1) \\ \quad {\scriptstyle = BS^1} \uparrow & & \in \\ \{*\} & \longmapsto & \dim C \end{array}$$

So $\underline{\dim C}$ must have an action of S^1.

But we broke the circle into two parts: ie we only computed $\dim C$ <u>non</u>-equivariantly.

Can't necessarily take all $\mathbb{CP}^\infty$ to this one point $\dim \mathcal{C}$ — it may or may not, depending on the field theory!

Example k a field
$\mathcal{C}$ = category with objects = k-algebras, morphisms = classifying spaces of chain complexes of bimodules

Every object in $\mathcal{C}$ is dualizable!
$A \in \mathcal{C} \rightsquigarrow A^\vee = A^{op}$.

$\dim A = HH_*(A)$ relative to k Hochschild homology

– circle action comes from bar resolution... nontrivial!

Theory predicts dimensions of dualizable objects have S^1 actions!