J. Lurie - TFT in Low Dimensions III

Note Title 3/17/2008

Last time: 1Bord $(\infty,1)$ category
  Objects: 0-manifolds
  morphisms: classifying spaces for bordisms of 0-manifolds

symmetric monoidal under $\amalg$.

Theorem If $\mathcal{C}$ is any symmetric monoidal $(\infty,1)$-cat.
$\Rightarrow \mathrm{Fun}^{\otimes}(1\mathrm{Bord}, \mathcal{C}) \simeq$ dualizable objects $c \in \mathcal{C}$

$$F \longmapsto F(\overset{+}{*})$$

Consider the special case: $\mathcal{C}$ is a Picard $(\infty-)$groupoid: all morphisms are invertible & all objects invertible wrt $\otimes$ — ie $\mathcal{C}$ is an $(\infty,0)$-category $\Leftrightarrow$ a space X, with a "comm. assoc." multiplication up to coherent homotopy: X is in fact an infinite loop space, or spectrum.

1Bord is not a Picard groupoid, but any map to $\mathcal{C}$ as above factors through completion of 1Bord where we invert

all morphisms, aka a topological space
$||1\mathrm{Bord}||$ ... geometric realization of the
simplicial space underlying $1\mathrm{Bord}$
as a complete Segal space.

$$\mathrm{Fun}^{\otimes}(1\mathrm{Bord}, \mathcal{C}) \simeq \mathrm{Fun}^{\otimes}(||1\mathrm{Bord}||, \mathcal{C})$$

Thm $/S$ $\simeq \mathrm{Hom}_{\substack{\infty\text{ loop}\\ \text{spaces}}}(||1\mathrm{Bord}||, X)$

$X =$ all (dualizable) objects in $\mathcal{C}$

ie $||1\mathrm{Bord}||$ has universal property among infinite loop spaces: Theorem in this case just says $||1\mathrm{Bord}|| \simeq QS^0 \stackrel{\text{def}}{=} \varinjlim \Omega^n S^n$

the stable sphere: consequence of the work of Galatius-Madsen-Tillman-Weiss in case $n=1$, computing homotopy type of bordism categories.

<u>2Bord</u> $(\infty, 2)$ category: all morphisms above 2 are invertible.

$\otimes$ category under $\amalg$ (symm. monoidal).

Objects: 0-manifolds M, N

morphisms: bordisms of 0-manifolds B, B'

space of 2-morphisms:
$$\begin{tikzcd} M \arrow[r, bend left=50, "B"{name=U}] \arrow[r, bend right=50, "B'"'{name=D}] & N \arrow[from=U, to=D, phantom, "\Downarrow"] \end{tikzcd}$$

$2\mathrm{Hom}(B, B')$ = classifying space for bordisms from B to B', trivial along M & N.

— probably can't strictify, need more systematic definition in flavor of complete Segal spaces (as bisimplicial topological space).

<u>Question</u> $\mathcal{C}$ another symm. monoidal $(\infty,2)$ category
- describe $\mathrm{Fun}^{\otimes}(2\mathrm{Bord}, \mathcal{C})$

Easier version: case $\mathcal{C}$ is a Picard groupoid, in this case we're calculating $|2\mathrm{Bord}|$ as an infinite loop space (solved by G-M-T-W).

Theorem (GMTW) $|2Bord| \simeq \Sigma^2 MTSO(2)$

ie 2Bord has a presentation as the cofiber of a map $(QS^1)_{S^1} \to QS^0$

("cokernel") : generators coming from $QS^0 = |1Bord|$. 2Bord has 1Bord inside & these are generators, so need to impose some relations: kill off QS^1 :

$QS^1 = \varinjlim \Omega^n S^{n+1}$ "one relation in deg one" :

the circle dies in 2Bord since it bounds ... in particular it bounds the disc. The rotation action of the circle extends to the disc: S^1 bounds S^1-equivariantly! so one generator & one relation.

Goal: give an analogous presentation of 2Bord itself by generators & relations –

try to lift GMTW to a presentation of 2Bord itself.

Suppose $F : 2\mathrm{Bord} \longrightarrow \mathcal{C}$ 2d TQFT.
What do we get out of this?

$F(\overset{+}{*}) = C \in \mathcal{C}$ dualizable (dual: $F(\overset{-}{*})$)

$\rightarrow$ determines $F(\bigcirc) =$ "dim C" $\in \mathrm{Hom}_{\mathcal{C}}(1,1)$.

$$F(\bigcirc) = \eta \in 2\mathrm{Hom}_{\mathcal{C}}(id_1, \dim C)$$

dr = bordism $\emptyset \to S^1$ — not an isomorphism, just a morphism from id to dim C.

The whole point of the description of 1Bord is that dim C has an action of S^1

– above map is in fact <u>equivariant</u>

$$\eta \in 2\mathrm{Hom}_{\mathcal{C}}(id_1, \dim C)^{S^1}$$

<u>Theorem</u> $\mathrm{Fun}^{\otimes}(2\mathrm{Bord}, \mathcal{C}) \simeq \{ (C, \eta) : C \in \mathcal{C}$ dualizable, $\eta \in 2\mathrm{Hom}(id_1, \dim C)^{S^1}$ "nondegenerate" $\}$

nondegeneracy: analog for 2-Bord of dualisability of $\mathcal{C}$. not extra data, just a condition

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$2\mathrm{Bord}^0$: objects: 0-manifolds
morphisms: bordisms
2-morphisms: $2\mathrm{Hom}(B,B') =$
$\left\{ \text{surfaces } \Sigma \text{ s.t. each component of } \Sigma \text{ meets } B' \right\}$

ie  ok  not ok

<u>Thm</u> $\mathrm{Fun}^{\otimes}(2\mathrm{Bord}^0, \mathcal{C}) \simeq$ same RFS with easier nondegeneracy condition.

<u>2-categories</u>
Example: Cat: obj: categories
morphisms: functors
2-morphisms: natural transformations

Idea: notion of adjoint functors

$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ is an <u>adjunction</u> if we have canonical isomorphisms

(*) $\mathrm{Hom}_{\mathcal{C}}(C, GD) \simeq \mathrm{Hom}_{\mathcal{D}}(FC, D)$

$\forall C \in \mathcal{C}, D \in \mathcal{D}$.

A definition in purely 2-categorical terms:
if we have an adjunction, can take

$$id_{GD} \in \mathrm{Hom}_{\mathcal{C}}(GD, GD) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(FGD, D)$$

$\Rightarrow$ natural transformation $v: F \circ G \longrightarrow id_{\mathcal{D}}$
which recovers our isomorphism (*):

$$\mathrm{Hom}_{\mathcal{C}}(C, GD) \xrightarrow[F]{} \mathrm{Hom}_{\mathcal{D}}(FC, FGD) \xrightarrow{v} \mathrm{Hom}_{\mathcal{D}}(FC, D).$$

To see that this is invertible need map in other directions: unit $u: Id_{\mathcal{C}} \to G \circ F$,
+ compatibility with v.

Can make sense of adjunctions in any $(\infty, 2)$ category: Given objects C, D &
morphisms $f: C \to D$ & $g: D \to C$
can ask if these are adjoints,

ie $\exists$ unit map $v: I_C \longrightarrow g\circ f$
& a counit $v: f\circ g \longrightarrow I_D$, compatible.
(space of counits for which there is a unit).

If $C \in \mathcal{C}$ dualizable $\Rightarrow \exists C^\vee$,

& $\underline{1} \xrightarrow{coev_C} C \otimes C^\vee \xrightarrow{ev_C} \underline{1}$

$\dim C = ev \circ coev$

ie have $\eta: id_{\underline{1}} \longrightarrow ev \circ coev$.

Can ask if this natural transformation is an adjunction.

<u>Theorem</u> $Fun^{\otimes}(2Bord^{\circ}, \mathcal{C}) \simeq \{ C, \eta : C \in \mathcal{C}$ dualizable
$\eta \in 2Hom(id_1, \dim C)^{S^1}$ st
η is the unit of an adjunction $\}$

These nondegeneracy conditions force into existence lots of objects from above:
e.g. C dualizable $\Rightarrow \exists C^\vee$,
$\exists\ 1 \rightleftarrows C \otimes C^\vee$, & above have counit, ...

What is the counit? coev ∘ ev → id.

Draw in 2Bord:

$\emptyset$

2 morphism

η  disc

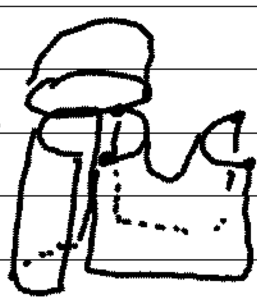
ev ∘ coev

1 morphism

 coev ∘ ev

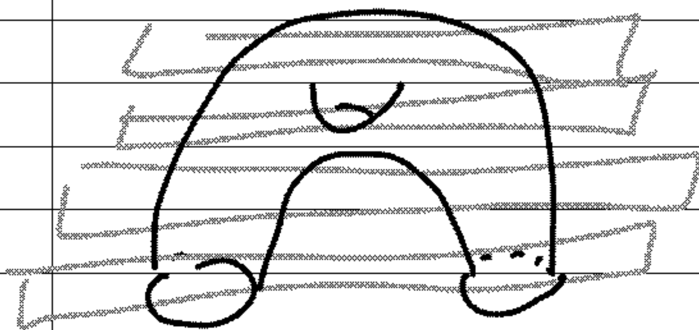
= id.

What is compatibility of unit & counit?

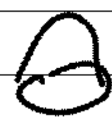
η  $\rightsquigarrow$ 

2-morphisms in 2Bord^0:

given surface cut into pieces using a Morse function



Only need Morse indices 0,1 in 2Bord^0

– only need  &  saddle

(in full 2Bord would need also )

So given Morse function can decompose into the operations we had before.

Any 2 Morse functions are related by basic moves as above (compatibility of unit & counit) — Morse/Cerf theory

To get $(\infty,2)$ categorical info however would need to use arbitrary dimension families of Morse functions & their singularities....

Actual proof: use ribbon graph model for moduli spaces of Riemann surfaces, following work of K. Costello.

Framed analog:

$$\mathrm{Fun}^{\otimes}(2\mathrm{Bord}^{fr}, \mathcal{C}) \simeq \text{Objects } c \in \mathcal{C} + \text{nondegeneracy}$$

... ie $2\mathrm{Bord}^{fr}$ is a free sym monoidal $(\infty,2)$-category on a nondegenerate object....

[Baez-Dolan conjecture]

Replacing $2\mathrm{Bord}^{0}$ by $2\mathrm{Bord}$: need another nondegeneracy condition, involving

as well --- corresponds to demanding certain maps are adjoint.

Next time: how to apply this in practice? need • & trace on algebra

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