J. Lurie - Topological Field Theory in Low Dimensions IV

Note Title 3/18/2008

<u>Last time</u>: symmetric monoidal $(\infty,2)$-category

$$2\mathrm{Bord}\begin{cases}\text{objects: 0-manifolds}\\ \text{morphisms: bordisms of 0-manifolds}\\ \text{2-morphisms: classifying spaces for bordisms of bordisms.}\end{cases}$$

Let T be another symmetric monoidal $(\infty,2)$-category (target)

2d TFT valued in T:

$$\mathrm{Fun}^{\otimes}(2\mathrm{Bord}, T) = \left\{ \begin{array}{l} (C,\eta): C\in T \text{ is a dualizable object} \\ \eta \in 2\mathrm{Hom}_T(\mathrm{id}_{1}, \dim C)^{S^1} \\ \text{"nondegenerate"} \end{array} \right\}$$

- Similar theorem for $2\mathrm{Bord}^{\circ} \subset 2\mathrm{Bord}$, weaker nondegeneracy (not allowed to cap off)

What are examples of higher categories?

(1.) Classical Homotopy theory: $\{(\infty,0)\text{-categories}\} \Longleftrightarrow \{\text{topological spaces}\}$

- study of this particular $(\infty,1)$ category

Modern homotopy theory (since Quillen):
study <u>other</u> $(\infty,1)$ categories.

eg · stable homotopy theory: study of spectra

· rational " " : spaces up to rationalization

· ...

(2) "Deloopings":

· A category $\mathcal{C}$ with one object (distinguished)
$\iff$ an associative monoid

· A 2-category w/ 1 distinguished object
$\iff$ a monoidal category (associative $\otimes$ product)

· etc etc. : can trade associative multiplications for higher categoricity.

(3) Categories of categories:
$\{$All (n,m)-categories$\}$ forms an $(n+1, m+1)$ category
many variations — categories with additional structure.

Today: K a field.

$dgCat_K$: an $(\infty,2)$ category

- objects: differential graded categories / K: Homs are chain complexes / K
- Morphisms: morally dg functors
- higher morphisms: natural transformations ... or rather the underlying chain complexes

- really need to consider functors that only preserve composition up to coherent homotopies.

we'll assume our dg categories admit homotopy colimits & functors commute with them
$\Rightarrow$ have infinite $\oplus$, have cones etc;
implies the homotopy categories are triangulated
ie roughly objects of dgCat give triangulated categories

$dgCat_K$ has a $\otimes$ product:

incorrect def: $\mathcal{C} \otimes \mathcal{D}$ has objects $C \otimes D$, $C \in \mathcal{C}$, $D \in \mathcal{D}$, morphisms
$Hom(C \otimes D, C' \otimes D') = Hom(C,C') \otimes Hom(D,D')$

Wrong definition: need "linear combinations" of pure tensors to get good tensor product ... ie need to allow homotopy colimits.

Better definition: $\mathcal{C}, \mathcal{D} \rightsquigarrow \mathcal{C} \otimes \mathcal{D}$ has the following universal property

$$\mathrm{Fun}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \text{Multilinear functors } (\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

... preserve colimits separately in each variable.

This $\otimes$ makes dgCat into a symmetric monoidal $(\infty, 2)$ category. Unit object $\underline{1} \in \mathrm{dg\,Cat}_K$ is {chain complexes $/K$} ie dgVect_K.
(unbounded in both directions)

More generally: for A a K-algebra
Mod_A = dg category of injective complexes of left A-modules

(more generally, A can be a dg K-algebra)

$$\mathrm{Mod}_A \otimes \mathrm{Mod}_B \simeq \mathrm{Mod}_{A \otimes B}$$

Observe: $\forall A$, Mod_A is dualizable, dual is $\mathrm{Mod}_{A^{op}}$

Proof: need

$$\begin{tikzcd}
1 \arrow[r] \arrow[dr, "\otimes_k A"'] & \mathrm{Mod}_A \otimes \mathrm{Mod}_{A^{op}} \arrow[r] \arrow[d, phantom, "\wr"] & 1 \\
& \mathrm{Mod}_{A \otimes A^{op}} \arrow[ur, "\otimes^L_{A \otimes A^{op}} A"'] &
\end{tikzcd}$$

$\boxtimes$

Composition $1 \longrightarrow 1$ is given by tensor by a particular dg vector space:

$$\dim \mathrm{Mod}_A = A \overset{L}{\underset{A \otimes A^{op}}{\otimes}} A = HC_*(A)$$

Hochschild chains on A.

.... note have interesting circle action on $\dim \mathrm{Mod}_A$, Connes' cyclic differential B.

Example String topology.

M a (compact oriented) manifold.

Define $Loc(M) \in dgCat_k$

$Loc(M)$ = dg category of (injective) complexes of sheaves of k-vector spaces on M with locally constant cohomology

--- roughly local systems of chain complexes: parallel transport is a quasi-isomorphism, defined up to chain homotopy.

Remark: If M is connected, & we choose point $x \in M \Rightarrow$ consider based loops at x, ΩM is a monoid,

$C_*(\Omega M, k)$ is a dg algebra$/k$

& $Loc(M) \simeq Mod_{C_*(\Omega M; k)}$:

"modules over the group ring of the fundamental group"

Loc(M) is self dual:

$$\begin{tikzcd}[column sep=small]
& & LM = \mathrm{Map}(S^1, M) \arrow[dl] \arrow[dr] & & \\
& M \simeq PM \arrow[dl] \arrow[r, "\Delta"] & M \times M & PM \simeq M \arrow[l] \arrow[dr] & \\
* & & & & *
\end{tikzcd}$$

$$\begin{array}{ccccc}
\mathrm{Loc}(*) & & \mathrm{Loc}(M \times M) & & \mathrm{Loc}(*) \\
\wr & & \wr & & \wr \\
1 & \longrightarrow & \mathrm{Loc}\, M \otimes \mathrm{Loc}\, M & \longrightarrow & 1
\end{array}$$

apply pullback & pushforward:
need to replace $M \longrightarrow M \times M$
by a fibration to get the pushforward
to be locally constant $\rightsquigarrow$

replace $M \simeq PM = \mathrm{Map}([0,1], M)$

$$\begin{tikzcd}
& M \simeq PM \arrow[dl] \arrow[dr, "\text{fibration}"] & \\
* & & M \times M
\end{tikzcd}$$

pushforward will be homology not cohomology
along fibers: need to preserve
colimits.

By push-pull identity we find that the composition of the maps is given by passing through the loop space:

$$\dim_{Loc(M)} \simeq C_*(LM; k)$$

$\circlearrowleft_{S^1}$ rotate loops

$\rightsquigarrow$ Field theory with $S^1 \mapsto$ homology of LM.

To extend to a 2d TFT need to specify a trace η :

$$\begin{array}{ccc} id_1 & \longrightarrow & \dim C \\ \wr\!\!| & & \wr\!\!| \\ k & \longrightarrow & C_*(LM; k) \end{array}$$

ie need a cycle in the homology of the loop space, & should be S^1 equivariant: try to get it from M itself

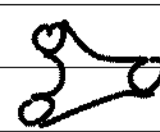
$M = (LM)^{S^1} \subset LM$: idea take η to be the pushforward of a nice cycle on M

M a manifold $\longrightarrow$ have <u>fundamental cycle</u>, take η = pushforward of the fundamental cycle.

η is nondegenerate $\Longleftrightarrow$ M satisfies Poincaré duality.

<u>Corollary</u> Get a field theory

$$F : 2\mathrm{Bord}^{\circ} \longrightarrow dg\,\mathrm{Cat}_k$$
$$F(\bigcirc) = C_*(LM; k)$$

.... giving string topology:  on homology gives Chas-Sullivan product on $H_*(LM)$.

<u>Corollary</u> String topology is a homotopy invariant notion --- defined for any k-Poincaré duality space.

.... literally above definitions require η to be a 0-cycle, ie $\dim M = 0$!

if $\dim M > 0$ need to replace dgCat_k by $\widetilde{\mathrm{dgCat}}_k = \{ (\alpha, \mathcal{C}) :$ α is a 2-gerbe, $\mathcal{C}$ is an α-twisted dg category over $k \}$

– α component gives just an integer, which is giving a shift $\rightsquigarrow$ twisted version of above story in which η is no longer a 0-cycle.

(Space of 2-gerbes is connected – i.e. have only one such – but not contractible, so gives twisting of action of category)

– motivates more abstract targets, so as to encode various twisted versions?

Higher dimensions

Let's simplify the theorem in two ways:

- instead of stating nondegeneracy, work in context where they are automatically satisfied (idea of Baez-Dolan): restrict to setting where T "has duals" (condition on T)

  (A) If $n=1 \Rightarrow$ this means every object of T is dualizable

  (B) If $n=2 \Rightarrow$ (A) + every morphism in T should have left & right adjoints.

Rmk 1. - these conditions are very strong!
dgCat$_k$ doesn't satisfy it; but have always a largest subcategory which does satisfy it
dgCat$_k$: (A) is satisfied by e.g

QCoh(X) if X is quasiprojective
(X has enough vector bundles e.g)

(B) is satisfied by QCoh(X) $\Leftrightarrow$
X is smooth & proper

$n \geq 3$ conditions get even stronger!

[Hopkins proposal for examples: instead of numbers at top degree get elements in Lubin-Tate deformation rings.]

2. The conditions are harmless, in the sense that $n\mathrm{Bord}$ has duals, so any TFT will land in the subcategory of things with duals in the full n-dimensional sense!

So let's assume $\mathcal{T}$ has duals.

Why was S^1 appearing in the statement?

- came from structure group of our manifolds — so let's get rid of this:

$n\mathrm{Bord}^{fr}$: uses framed n-manifolds ... trivializing all tangent bundles in dimension n (e.g. for a point trivialize $\mathbb{R}^n$)

Conjecture (Baez-Dolan cobordism hypothesis)
Assume T has duals

$$\mathrm{Fun}^{\otimes}(n\mathrm{Bord}^{fr}, T) = \{\text{objects of } T\}$$

... i.e. $n\mathrm{Bord}^{fr}$ freely generated by a single object

Now suppose we want manifolds with a different structure group:

$O(n) \circlearrowright n\mathrm{Bord}^{fr}$ change of framing

$\Rightarrow$ conjecture implies $\{\text{objects of } T\} \circlearrowleft O(n)$

e.g. $n=1$: $O(1)$ action on category with duals, $V \longmapsto V^*$

eg T Picard groupoid $\Rightarrow$
$(T)_{\text{gth space}} \circlearrowleft O(n)$: J-homomorphism from homotopy theory

So if $G \longrightarrow O(n)$ get a category

$n\mathrm{Bord}^G$ (eg $G = SO(n)$: oriented manifolds)

$\Rightarrow$ Conjecture $\mathrm{Fun}^{\otimes}(n\mathrm{Bord}^G, \mathcal{T}) = \{\text{obj of } \mathcal{T}\}^G$

homotopy fixed points

(Optimistically – soon theorem!
looks like there's a proof in any
dimension [Hopkins - Lurie])

Proof in higher dimensions very different in flavour!

2-Gerbe case: first specify theory valued
is space of 2-gerbes : $\{2\text{-gerbe}\}^{SO_2}$

– make trivial 2-gerbe SO_2 - fixed in
nontrivial way : π_0 of ways to do that
are exactly $\mathbb{Z}$ $\rightsquigarrow$ notion of shift.

Discussion

Example of n-categories with duals:

$n=1$ Ob = algebras
1-Mor = bimodules
higher = isomorphisms

If we make this into a 2-category by allowing morphisms between bimodules then this category no longer has duals: need our bimodules to have adjoints, so we restrict to those, but then our objects are no longer dualizable since eval & coeval maps to/from unit aren't allowed anymore $\longrightarrow$ must restrict to saturated algebras (or dg categories) i.e. smooth & proper — exactly corrects eval & coeval, diagonal is now perfect etc.

Higher example: for any n we have an n-category with duals:
Ob: E_n-algebras
Morph(A, B) = E_{n-1} algebras C with

E_n-maps $\quad A \to Z_{E_{n-1}}(C) \leftarrow B$

2-morphisms: E_{n-2}-algebras,

$\circ \quad \bullet \circ$

n-morphisms: spectra

$(n+1)$- " : isomorphisms.

<u>But</u> if we relax $n+1$ morphisms
to be morphisms of spectra
.... not $(n+1)$ category with duals any more!
Hard to find examples of objects with duals in here!

<u>Mike's idea</u> Fix $m \geq 0$, work in the $K(m)$-local stable homotopy category,
$\mathcal{E}_m$ = Lubin–Tate spectrum (which is p-adically complete)

Consider a p-finite space X.
let $A = \hat{C}_*(\Omega^n X, \mathcal{E}_m)$
Completed chains on $\Omega^n X$ with Lubin-Tate coefficients

[eg should take $m = n$ $X = K(n, \mathbb{Z}/p)$]

.... this seems to give a good finite example of a field theory ..

- the problem is we need a space which is finite & whose loop space is finite... but here one can hope to take advantage of some of the surprisingly strong finiteness properties of E_m-cohomology..