Alina Marian - Strange Duality for Surfaces

Note Title 4/30/2008

S surface $\Rightarrow$ speculations of le Potier (late 90s) on numerical & geometric strange duality
- duality of spaces of sections (level / rank)

Hard to compute holomorphic Euler characteristic
$\chi(\mathcal{M}_v, \Theta_w)$ .. K-theoretic Donaldson invariants (Göttsche - Nakajima - Yoshioka)
--- Donaldson invariants twisted by Todd genus.

Take L line bundle on S surface, $\chi(L) = n = h^0(L)$

take first topological type $v = [I_Z]$ ideal sheaf with $\ell(Z) = k < n$

second topological type $w = [J_W \otimes L]$ $\ell(W) = n-k$

$\mathcal{M}_v = S^{[k]}$ $\mathcal{M}_w = S^{[n-k]}$

On $S^{[k]}$ have determinant line bundle

$$L^{[k]} = \det R p_* (\underbrace{\mathcal{O}_{\mathcal{Z}}}_{\text{universal subscheme} \subset S^{[k]} \times S} \otimes q^* L)$$

$$H^0(S^{[k]}, L^{[k]}) = \Lambda^k H^0(S, L) \; :$$

follows from considering Hilbert-Chow morphism to $S^{(k)}$.

Divisor $\Theta_{k,n} = \{ (I_Z, I_W) : h^0(I_Z \otimes I_W \otimes L) \neq 0 \}$
$\subset S^{[k]} \times S^{[n-k]}$

$$\mathcal{O}(\Theta_{k,n}) = L^{[k]} \boxtimes L^{[n-k]}$$

$$\begin{array}{ccc} D : H^0(S^{[k]}, (L^{[k]})^\vee & \longrightarrow & H^0(S^{[n-k]}, L^{[n-k]}) \\ \| & & \| \\ \Lambda^k H^0(S,L)^\vee & \simeq & \Lambda^{n-k} H^0(S,L) \end{array} \quad \checkmark$$

Now assume S is a K3 surface.

For E on S let $v(E) = ch\, E \sqrt{td_S} \in H^{ev}(S, \mathbb{Z})$
(Mukai vector) $= v_0 \oplus v_2 \oplus v_4$

Mukai pairing : $\langle v, w \rangle = \int_S v_2 w_2 - v_0 w_4 - v_4 w_0$

Fix v primitive in $H^{ev}(S, \mathbb{Z})$ & $v_0 > 0$
(i.e consider positive rank sheaves).

$\Rightarrow$ $\mathcal{M}_v$ is smooth, consists only of stable sheaves, & has dimension $\langle v,v\rangle + 2$,

Properties 1. $\mathcal{M}_v$ has an irreducible holomorphic symplectic structure (so if dim $= 2 \Rightarrow$ K3 again)

2. $\mathcal{M}_v$ is deformation equivalent to $S^{[d_v]}$
$d_v = \frac{1}{2}\langle v,v\rangle + 1$ Hilbert scheme
(O'Grady - Yoshioka)

3. There is a bilinear form on $H^2(\mathcal{M}_v, \mathbb{Z})$
- Beauville - Bogomolov form,

$$K(S) \supset v^{\perp} \longrightarrow H^2(\mathcal{M}_v, \mathbb{Z})$$
$$w \longmapsto c_1(\theta_w)$$
$$(w,w) = B(c_1(\theta_w)) \quad \text{B-B form}$$

4. $\chi(\mathcal{M}_v, \mathcal{L})$ is a deformation invariant polynomial in $B(c_1(\mathcal{L}))$ (for any line bundle $\mathcal{L}$) $\Rightarrow$ can calculate on Hilbert scheme!

$$\chi(\mathcal{M}_v, \theta_w) = \binom{d_v + d_w}{d_v}$$
$$= \chi(\mathcal{M}_w, \theta_v) \checkmark$$

$d_v = \frac{1}{2}$ dim of moduli space

When S is an abelian surface, have $\mathcal{M}_v$

$$\det : \mathcal{M}_v \longrightarrow \hat{S} \times S$$
$$E \longmapsto \det E \quad \det \mathcal{F}(E)$$

Fourier-Mukai transform

Fiber of this map $= K_v$ (Kummer)

- irred hol symplectic

$$\Rightarrow \quad \chi(K_v, \Theta_w) = \chi(\mathcal{M}_w, \Theta_v)$$

numerical strange duality.

[Geometric strange duality works for elliptically fibered K3s.]