Kevin McGerty - Hall Algebras & Quantum Frobenius

Note Title 4/15/2008

1. Modified Quantum Groups
2. Quantum Frobenius
3. Hall algebras
4. connect 2 & 3

A root datum is a pair X, Y of free abelian groups $\langle\ \rangle : Y \times X \longrightarrow \mathbb{Z}$ perfect pairing & simple roots $\{\alpha_i\} \subset X$ $\{\alpha_i^\vee\} \subset Y$ coroots ($i \in I$), a symmetric pairing $\cdot : \mathbb{Z}I \times \mathbb{Z}I \to \mathbb{Z}$ such that $\frac{2 i \cdot j}{i \cdot i} = \langle \alpha_i^\vee, \alpha_j \rangle$ a generalized Cartan matrix (a_{ij})

$\Rightarrow$ quantum group $\mathcal{U}$, an algebra over $\mathbb{Q}(v)$, generated by $\{E_i, F_i, K_\mu : i \in I, \mu \in Y\}$ & subject to a "quantized Serre presentation"

e.g. $\displaystyle\sum_{s=0}^{1-a_{ij}} (-1)^s E_i^{(s)} E_j E_i^{(1-a_{ij}-s)} = 0$

where $E_i^{(s)} = E_i^s / [s]_i!$,

$[s]_i! = [s]_i [s-1]_i \cdot \ldots [1]_i$

& $[k]_i = \dfrac{v_i^k - v_i^{-k}}{v_i - v_i^{-1}}$ $\qquad v_i = v^{\frac{1}{2}(i \cdot i)}$

Work with a modified form of U:

Let Mod_X be the category of reps of U with a weight decomposition, $V = \bigoplus V_\lambda$

$V_\lambda = \{ w \in V \mid K_\mu w = v^{\langle \mu, \lambda \rangle} w \}$

Let $\hat{U} = \mathrm{End}(\mathrm{Forget} : \mathrm{Mod}_X \longrightarrow \mathrm{Vect})$

Inside $\hat{U}$ have projection operations - project to λ-weight space.

Set $\dot{U} = \bigoplus_{\lambda \in X} U \cdot 1_\lambda \subset \hat{U}$, a $\mathbb{Q}(v)$-subalgebra of $\hat{U}$.

- nonunital algebra but have lots of idempotents; can look at unital modules for $\dot{U}$, get Mod_X.

Integral forms: If $U^+ = \langle E_i \rangle \subset U$, define an $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ form (Lusztig)

$U^+_{\mathcal{A}} = \mathcal{A}$-algebra inside U^+ generated by $\{ E_i^{(n)} : i \in I, n \geq 1 \}$.

$\dot{U}_{\mathcal{A}}$ = the $\mathcal{A}$-subalgebra of $\dot{U}$ generated by
$\{E_i^{(n)} 1_\lambda,\ F_i^{(n)} 1_\lambda : i \in I,\ \lambda \in X, n \geq 0\}$

<u>Theorem</u> $\dot{U}_{\mathcal{A}}$ is an integral form of $\dot{U}$.

<u>Remark</u> $\dot{U}_{\mathcal{A}}$ has a canonical basis $\dot{B}$ which is a $\mathbb{Q}(v)$ basis of $\dot{U}$

- yields canonical bases of tensor products (highest weight modules V) $\otimes$ (lowest weight module V) when this makes sense

- if B is the canonical basis of U^+ $\Rightarrow$ $b 1_\lambda \in \dot{B}$ $\forall b \in B$.

Roots of unity: Let $\mathcal{A}_\ell = \mathcal{A} / (\Phi_{2\ell}(v))$ where $\Phi_{2\ell}$ is the 2ℓ-th cyclotomic polynomial

Can specialize $\dot{U}_{\mathcal{A}}$ to $\mathcal{A}_\ell$, write $\dot{U}_\ell = \mathcal{A}_\ell \otimes_{\mathcal{A}} \dot{U}$

" $\dot{U}_\ell$ behaves like an algebraic group in characteristic ℓ" : if ℓ is prime, we may base change $\dot{U}_\ell$ to a field of characteristic ℓ sending $v \mapsto 1$ & the resulting algebra is a modified form of the hyperalgebra of the associated group (root datum of finite type)

(hyperalgebra = distributions supported at the identity)

Quantum Frobenius : The Frobenius on the hyperalgebra lifts to $\dot{U}_\ell$ as follows:

(assume ℓ coprime to all $\{\frac{1}{2}(i \cdot i),\ i \in I\}$)

Theorem (Lusztig) There is a surjective algebra map

$$Fr_q : \dot{U}_\ell \longrightarrow \dot{U}_1 \underset{\mathbb{Z}}{\otimes} \mathcal{A}_\ell, \text{ characterized}$$

by $Fr_q(E_i^{(n)} 1_\lambda) = \begin{cases} E_i^{(n/\ell)} 1_\lambda & \text{if } \ell \mid n \\ 0 & \text{otherwise} \end{cases}$

(same for $F_i^{(n)} 1_\lambda$)

Remark $\dot{U}$ is a bimodule for U^+, U^-
$\mathbb{1}$ can be defined in terms of these actions.
So if you know the existence of an analogue of
Fr on U^+ it is straightforward to "extend"
to $\dot{U}_\ell$.

Hall algebras

Let $Q = (J, H)$ be a quiver, vertices J & edges H.

- Orientation $\Omega \iff H \underset{t}{\overset{s}{\rightrightarrows}} J$
- Automorphism $\underline{a}$ of the oriented quiver
s.t. if $h \in H \implies s(h), t(h)$ lie in different
a-orbits of J
$I :=$ set of a-orbits

$\mathcal{C}$ = category of reps of $\underline{Q}$ with objects V:
J-graded vector spaces $/\overline{\mathbb{F}_q}$ + Frobenius map
F, $F(V_i) = V_{a(i)}$ giving an $\mathbb{F}_q$-structure.

$V \in \mathrm{Ob}\,\mathcal{C}$, set $E_V = \bigoplus_{h \in H} \mathrm{Hom}(V_{s(h)}, V_{t(h)})$

$G_V = \prod_{i \in I} GL(V_i)$

$\mathcal{H}_V$ = $\mathbb{Z}$-valued G_V^F-invariant functions on E_V^F

– depends only on graded dimension v of V

Let $\mathcal{H} = \bigoplus_{v \in \mathbb{N}I} \mathcal{H}_v \implies$ an algebra via convolution

$$f * g(x) = \sum_{\substack{W \subset V \\ x(W) \subset W}} f(x|_W)\, g(x|_{V/W})$$

- Need to twist multiplication by a cocycle

  $m : \mathbb{Z}J \times \mathbb{Z}J \longrightarrow \mathbb{Z}$

  $m(\nu, \mu) = \sum_{i \in J} \nu_i \mu_i + \sum_{h \in H} \nu_{s(h)} \nu_{t(h)}$

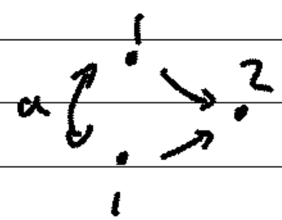
then twist $(\mathcal{H}, *)$ by $q^{-\frac{1}{2} m(\nu, \mu)}$

ie $f \cdot g(x) = q^{-\frac{1}{2} m(\nu,\mu)} f * g$ [need to introduce $q^{\frac{1}{2}}$]

• Take the subalgebra $\mathcal{C} \subset \mathcal{H}$ generated by $\{ 1_{n_i} \;\; n \in \mathbb{N} \;\; i \in \bar{I} \}$ constant function on E_V^F where $\dim V = n_i \in \mathbb{N}\bar{I}$

Can attach to (Q, a) a Cartan datum $(\mathbb{Z}\bar{I}, \cdot)$
where $i \cdot j = -\#\{\text{edges } h : s(h) \in i,\ t(h) \in j\}$ $\quad i \neq j$

$i \cdot i = 2 \cdot \#$ vertices in orbit i

eg  $\rightsquigarrow \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix} \rightsquigarrow B_2$

<u>Theorem</u> (Lusztig; Ringel / Green for finite type)

Let U_v^+ be the positive half of U attached to $(\bar{I}, \cdot)$. Then $U_v^+|_{v = q^{\frac{1}{2}}} \xrightarrow{\sim} \mathcal{C}$

where $E_i^{(n)} \longmapsto 1_{n_i}$.

Constructing the quantum Frobenius

Use the Hall algebra construction for $\mathbb{F}_q$ & $\mathbb{F}_{q^\ell}$

Let $\mathcal{C}$ be the Hall algebra wrt $\mathbb{F}_q$
$\mathcal{C}_\ell$ " " " " " $\mathbb{F}_{q^\ell}$

If V is a vector space in the $\mathbb{F}_{q^\ell}$-version
$\Rightarrow$ can forget $\mathbb{F}_{q^\ell}$-structure to get a vector space in the $\mathbb{F}_q$ version, say W.

Then there is a map $E_V \hookrightarrow E_W$

$\Rightarrow$ on Hall algebras/composition algebras get a restriction map $i^* : \mathcal{C} \longrightarrow \mathcal{C}_\ell$
- not an algebra map: counting $\mathbb{F}_q$ subspaces in $\mathcal{C}$ case & $\mathbb{F}_{q^\ell}$ subspaces in the $\mathcal{C}_\ell$ case.

To make the discrepancy disappear, use $\mathbb{F}_{q^\ell}^*$:

if $x \in E_v \hookrightarrow E_w$: embedding is described completely as taking fixed points for the action.

& if $\theta \in \mathcal{C}$, $\theta(x)$ is some count of subspaces in V & $\mathbb{F}_{q^\ell}^*$ acts on this set:

either a subspace is fixed (these we want to count) or its orbit has size $\frac{q^\ell - 1}{q^d - 1}$, in particular divisible by $\Phi_\ell(q)$.

$\Rightarrow$ change coefficients: mod out by $\Phi_\ell(q)$, get an algebra map

$$i^*: \mathcal{C} \bmod \Phi_\ell(q) \longrightarrow \mathcal{C}_\ell \bmod \Phi_\ell(q)$$

which (checking signs & twists) yields the quantum Frobenius: doing what we wanted on generators!

— exists mod infinitely many primes $\Rightarrow$ exists over the cyclotomic integers themselves!

Questions Lift to the level of stacks?

- replace the action of $\mathbb{F}_{q^\ell}/\mathbb{F}_q$ with the action of a cyclic group