D. Nadler - Langlands Duality for Real Groups

Note Title 3/14/2008

"Langlands-Vogan-Soergel-Mirror Duality"

(w/ D. Ben-Zvi)

Aim: Sketch a proof of Soergel's conjecture on the structure of real group representations

Ingredients:

Math	Physics
1. Langlands duality (Kazhdan-Lusztig, Bezrukavnikov) for affine Hecke algebras	S-duality (for certain surface operators)
2. S^1 equivariant localization	dimensional reduction
3. Base change (following Langlands)	"susy domain walls" (Witten, Gaiotto)

Do categorical linear algebra to build complicated categories from easy ones: theory of real groups follows from complex groups!

Ground rules: $G, G^\vee$ Langlands dual complex gps
Every category will be a derived, or really ∞, version

Real groups via TFT
(3d theory, maps into BG)

Worldsheets:
$\overset{B}{\bullet}\overset{G}{\text{———}}\overset{B}{\bullet}$:= G-bundle with reductions to Borel ("tame labels")

$= B\backslash G/B$ moduli space

--- controls complex group representation theory via Beilinson-Bernstein

Real group analog: $K \subset G$ fixed points of an involution (symmetric subgroup)
$\rightsquigarrow$ study $K\backslash G/B$. What picture shall we put? base change / folding

$\bullet\text{———}\bullet \longrightarrow \overset{B}{\bullet}\overset{G}{\text{——}}\times$

orbifold interpretation

$\overset{B}{\bullet}\xrightarrow{G}\overset{B}{\bullet}\ \overset{B}{\bullet}\xrightarrow{G}\times \longrightarrow \overset{B}{\bullet}\xrightarrow{G}\times$ action.

3d theory $\mathcal{X}_G$:

$\mathcal{X}_G(\bullet\!-\!\!-\!\bullet) = \mathcal{D}(B\backslash G/B)$ D-modules as a monoidal category

$\mathcal{X}_G(\bullet\!\longrightarrow\!\times) \sim \mathcal{D}({}_k\backslash G/B)$ (relation to be made precise later)

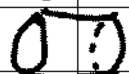
Spergel's conjecture, very roughly:

$\mathcal{X}_G(\bullet\!\longrightarrow\!\times) \sim \mathcal{X}_{G^\vee}(\bullet\!\longrightarrow\!\times)$

as module categories:

We'll understand the 3d theories via 4d theory
(Geometric Langlands)

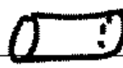
$\underline{A_G} \xleftrightarrow{\text{S-duality}} \underline{B_{G^\vee}}$



$\mathcal{D}(I\backslash LG/I) \longleftrightarrow \mathcal{O}(St_{G^\vee}/G^\vee)$ Steinberg

Bez.:
equivalent as monoidal categories

Real part of 4d theory:

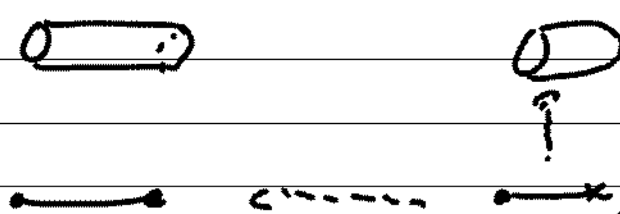
apply antipodal map to 

quotient:

 $\mathbb{RP}^2$ with a single boundary

Summary:

complex real

4D  $\mathbb{RP}^2$

3D $\mathbb{Z}/2$

access different parts of the story by going to 3d complex & 4d real.

Rapid review of representation theory of real groups

$$\begin{array}{ccc} G \text{ complex} & \supset & G_{\mathbb{R}} \text{ real form} \\ \cup & & \cup \\ K & \supset & K_{\mathbb{R}} \\ \text{complexification} & & \text{max cpt} \end{array}$$

Harish-Chandra: $G_{\mathbb{R}}$ - reps / inf. equivalence

$\longleftrightarrow$ $(\mathfrak{g}, K)$ - modules

$U\mathfrak{g} \supset Z$ Harish-Chandra center $\cong \mathbb{C}[\mathfrak{t}^\vee/W]$

Fix a regular integral character $\chi: Z \longrightarrow \mathbb{C}$
& work "near" χ throughout

<u>Beilinson-Bernstein localization</u>

$$(\mathfrak{g}, K)_\chi\text{-mod} = \mathcal{D}(K \backslash G/B) \circlearrowleft_{\mathcal{D}(B\backslash G/B)}$$

intertwiners

Allow generalized infinitesimal character:

$$(\mathfrak{g}, K)_{\hat{\chi}} = \mathcal{D}(K \backslash G/B_{\hat{\chi}})$$ relax notion of B-equivalence.

<u>Soergel conjecture</u> (Vogan duality, Langlands parametrization)
categorical K-groups sets

Fix an involution η of G (or really its inner class)
& dual involution $\eta^\vee$ of $G^\vee$
(involutions up to inner class $\longleftrightarrow$
involutions of Dynkin diagram)

<u>Conjecture</u> $\coprod_{\sigma\ \eta(\sigma)=1} \mathcal{D}(K_\sigma \backslash G / B_{\hat{\chi}})$ (whole collection of real forms

really $\exp(\rho^\vee)$

$$= \chi_G(\bullet \longrightarrow \times)$$

$$\updownarrow$$

$$\chi_{G^\vee}(\bullet \longrightarrow \times)$$

$$= \coprod_{\substack{\text{involutions} \\ \text{inner to } \eta^\vee}} \mathcal{D}(K^\vee_\sigma \backslash G^\vee / B^\vee)$$ strictly equivalent

Conjecture on mixed / Koszul equivalence

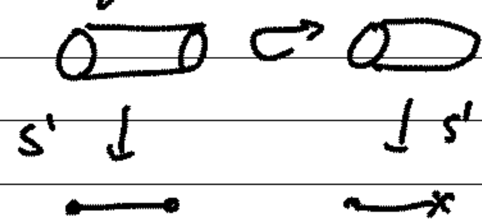
Note : on G side χ affects the category
on $G^\vee$ side it affects the space
but we chose regular integral χ so doesn't appear on RHS.

Rmk : Both sides are linear over $H^\vee$ = dual Cartan.

Theorem The conjecture holds for $\mathbb{Z}/2$ periodic versions

Witten comment: the parameters for 3d theory
(roughly) $\sim \mathbb{P}^3 / SL_2\mathbb{C} \supset$ generic point$/\mathbb{Z}/2$
(with maximal SUSY + dim reduction)

Ingredient 2 S^1 equivariant localization



A side

$D(I\backslash LG/I) \circlearrowleft_{S^1}$ loop rotation

Prop $D(B\backslash G/B)_{\text{periodized}}$
is
$D(I\backslash LG/I)^{S^1, per}$

– restrict cohomology calculations to fixed pts

B side

$\mathcal{O}(St^\vee/G^\vee) \circlearrowleft_{S^1}$ topological loops

Prop $St^\vee/G^\vee = \mathcal{L}(B^\vee\backslash G^\vee/B^\vee)$

$D(B^\vee\backslash G^\vee/B^\vee)$
is
$\mathcal{O}(St^\vee/G^\vee)^{S^1, per}$

A similar picture holds for the real parabolics

$\bigcirc \longrightarrow \bullet \longrightarrow \times$ on A, B side independently

$\Rightarrow$ A-side : $\coprod D(K_0 \backslash G / B)$

B-side : $\coprod D(K^\vee \backslash G^\vee / B^\vee)$

B side : easy to work upstairs

A side : " " " downstairs :

that's where the actions become tensor actions.

3. Base change : $\tilde{\Sigma} \xrightarrow[\Gamma]{} \Sigma$ covering

would like to compare theories on $\Sigma, \tilde{\Sigma}$.

What is the difference between

$B_{G^\vee}(\tilde{\Sigma}) \xleftarrow[(\Gamma^*)_*]{} B_{G^\vee}(\Sigma)$: pull back bundles

$\Rightarrow$ map on spaces, push forward coherent sheaves

Prop Ascent for this map is given by identifying the spherical Hecke action of Γ-conjugate points. $B_{G^\vee}(\Theta)$

– accessible since we can handle coherent sheaves more easily than D-modules

Conclusion Can recover $\coprod D(K^\vee \backslash G^\vee / B^\vee)$ from $\mathcal{O}(\mathfrak{St}^\vee / G^\vee)$ by S^1 + base change.
– ie complete control from categorical constructn.

Challenge: automorphic base change:
would like this proposition on the A-side.
– ie control $A_G(\Sigma)$ via $A_G(\tilde{\Sigma})$ with $A_G(S^2)$ action.

But: we can achieve this after S^1 localization: ie control
$\mathcal{Z}_G(\bullet\!\!\longrightarrow\!\!\times)$ using $\mathcal{Z}_G(\bullet\!\!\longrightarrow\!\!\bullet)$
& spherical action:

What remains of spherical action to $\mathcal{D}$-modules on BG

Trying to controls $\mathcal{D}(G)$ via $\mathcal{D}(BG)$
... easy to do such things for $\mathcal{O}$-modules

Can still pick out homotopical subsets using $\mathcal{D}$-modules – eg collections of involutions

$\mathcal{B}(\mathbb{RP}^2)$: connections with pole!
$\sqrt{\text{Steinberg}}$: take square root of g & identify B_1, B_2 in $St = (g, B_1, B_2)$)

i.e $\left\{ \delta \in G^L \setminus G^\vee,\ B^\vee \subset G^\vee \text{ s.t. } \delta^2 \in B^\vee \right\} / G^\vee$

$B \bullet\!\overset{G}{\text{——}}\!\bullet B$ $\quad \mathcal{X}_G(S^1)$ = Drinfeld center
= character sheaves