Roman Bezrukavnikov - Rational Cherednik

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Algebras & Miracles of Science

- From Quantum Cohomology to Canonical Bases

- QH^* - mysterious structure, often formulated in terms of a connection: small QH is a flat connection over H^2 with fiber H^* (de Rham world)

- Bridgeland stability conditions - should conjecturally be linked to QH^*. (Betti world)

  ⟿ t-structure + additional data.

- Canonical basis: basis of irreducibles associated to one of these t-structures, in representation theoretic context.

Remark: canonical bases depend on graded versions of our categories ⟿ weights.
- not treated in Bridgeland theory.

Rep theory	Geometry	Stability	QH
Category $\mathcal{O}$	$\mathcal{D}$-mod (G/B)	?	?
U_q-mod^0 (small q. group at root of unity) or principal block of restricted reps $U_0(\mathfrak{g})$ in char. p	$\mathrm{Coh}(T^*G/B)$	$\widetilde{\mathfrak{h}^* \setminus H_{\alpha,n}}$ affine root hyperplane complements	?
Reps of rational Cherednik algebra (rational DAHA) over k with char $k = p$	$\mathrm{Coh}(\mathrm{Hilb}^n \mathbb{A}^2)$	$\widetilde{\mathbb{C} - \underline{\mu}}$ some set of roots of unity	$QH^*(\mathrm{Hilb}^n \mathbb{A}^2)$ Okounkov-Pandharipande
Symplectic reflection algebras	$\mathrm{Coh}\ \mathrm{Hilb}^n \widetilde{\mathbb{A}^2/\Gamma}$		QH: Maulik-Oblomkov

Quiver varieties ......

$$\mathrm{Hilb}^n \mathbb{A}^2 = \left\{ \begin{array}{c} \text{ideals of codim } n \\ J \subset k[x,y] \end{array} \right\} / \sim$$

= pairs of commuting matrices w/ a cyclic vector

... a Nakajima quiver variety

res. of sings.

$$(\mathbb{A}^2)^n / S^n$$

$$\mathrm{Hilb}^n \mathbb{A}^2 = X_n \times \mathbb{A}^2$$

where $X_n \longrightarrow \mathbb{A}^{2n-2}/S_n$

Ideals with center of mass $= 0$.

<u>Theorem</u> (Haiman, Bridgeland-King-Reid)

$$D^b\,\mathrm{Coh}(X_n) \simeq D^b\,\mathrm{Coh}^{S_n}(\mathbb{A}^{2n-2})$$

- particular case of general conjecture: given finite group $\Gamma \curvearrowright M$ preserving top-degree nondegen form & $\tilde{M} \to M/\Gamma$ crepant resolution $\Rightarrow D\,\mathrm{Coh}\,\tilde{M} \simeq D\,\mathrm{Coh}^{\Gamma} M$

.... derived category of crepant resolution should not depend on resolution - eg same as for noncommutative resolution $D\mathrm{Coh}^\Gamma M$.

t-structures corresponding to these resolutions should be points at ∞ of stability manifolds, & equivalences should correspond to homotopy classes of paths between these ends...

Bezrukavnikov-Kaledin: conjecture is true if M is symplectic vector space V, ω & $\Gamma \circlearrowright (V, \omega)$.

- based on quantization in positive characteristic, inspired by modular Lie algebra rep. theory.

e.g. For Weyl algebra $W = k\langle x_i, y_i\rangle / [x_i, y_j] = \delta_{ij}$ have a big center $k[x_i^p, y_i^p]$ (char $k = p$) & W is Azumaya over the center - $k = \bar{k}$ almost endos of a vector bundle

X a smooth variety $/k$, $\mathcal{D}(X)$ PD/crystalline diff ops, $Z(\mathcal{D}(X)) = \mathcal{O}(T^*X^{(p)})$

& $\mathcal{D}(X)$ is Azumaya over $T^*X^{(p)}$

- Given M, ω algebraic symplectic variety $/k$
  Look for Frobenius-constant quantization: Poisson center ($=$ p^{th} power of functions) deforms trivially to $Z(\mathcal{O}^h(M))$.

  Can achieve such for quotient singularities V/Γ $\quad \Gamma \curvearrowright (V, \omega)$ (Bez.-Kaledin)
  & $\mathcal{O}^h$ is an Azumaya algebra over $X^{(p)}$.

This quantization preserves some features of char. 0 quantization. e.g. G/B is $\mathcal{D}$-affine in char. 0, & remains <u>derived</u> $\mathcal{D}$-affine in characteristic p

<u>Theorem</u> : $R\Gamma : D(\mathcal{O}^h\text{-mod}) \xrightarrow{\sim} \Gamma(\mathcal{O}^h)\text{-mod}$

$$\wr$$

$$\mathrm{Coh}\,(X^{(p)})$$

--- Azumaya algebra is split in formal neighborhood of the special fiber.

Point of view on $\mathcal{O}^h$ (Bez.-Finkelberg-Ginzburg):

$\mathcal{H}ilb \subset T^*(\mathfrak{gl}(V)\times V\,/GL(V))$

open piece of cotangent to an Artin stack

$\Rightarrow$ canconstruct quantization by quantum hamiltonian reduction, ie diffops on the stack

Can also consider twisted diffops for any character of GL:

$\mathcal{O}^h_c$ $\quad c\in\mathbb{Z}$ on $\quad \mathfrak{gl}\,V\times V\,/GL(V)$

(can twist Frobenius constant quantization by Pic ... deform line bundles & take their endomorphisms).

$\mathcal{O}^h_c$ depends only on c mod p.

<u>Remark</u> All $\mathcal{O}^h_c$ are pairwise equivalent Azumaya algebras, but the choice of equivalence $\mathcal{O}^h_{\bar{c}_1} \simeq \mathcal{O}^h_{\bar{c}_2}$ $\quad (\bar{c}_i \in \mathbb{F}_p)$

depends on lifts of $\bar{c}_i$ to $\mathbb{Z}$.

(Convenient to normalize splitting near special fiber for $\bar{c} = -\frac{1}{2} \in \mathbb{F}_p$, compatible with duality).

<u>Theorem</u> $\Gamma(\mathcal{O}^h_c) = A_c$ spherical rational Cherednik algebra

... deformation of $S_n \ltimes k[x_n, y_n]$

$A_c = e H_c e \quad e \in k[S_n]$ idempotent of the trivial rep.

Theorem $R\Gamma : D(\mathcal{O}^h_c\text{-mod}) \xrightarrow{\sim} D(A_c\text{-mod})$
iff $H_c e H_c = H_c$, i.e. $A_c \underset{\text{Morita}}{\sim} H_c$

For large p this holds iff
$\mathbb{F}_p \ni c \notin Q_n = \left\{ \frac{-i}{d} : 0 < i < d \leq n \right\}$ mod p

Prop Assume $\bar{c}, \bar{c}+1 \notin Q_n$,
$\mathcal{O}^h_{\bar{c}} \sim \mathcal{O}^h_{\bar{c}+1}$ $\Rightarrow$ get 2 t-structures on
the same category - & they coincide.

$A_c \underset{\text{Morita}}{\sim} A_{c+1} \subset D(\mathcal{O}^h_c\text{-mod}) = D(\mathcal{O}^h_{c+1}\text{-mod})$

Thus we get a collection of t-structures on
$D(\mathcal{O}^h\text{-mod})$ or $D_{X_n^0}(X_n)$ (formal nbhd of special fibre)
indexed by intervals in $\mathbb{Z} \setminus \{c \mid \bar{c} \in Q_n\}$

eg if $p \equiv 1 \bmod n! \Rightarrow$

$\{c : \bar{c} \in Q_n\}$ is all $\frac{i}{d}(p-1) + kp$

$\approx p\left(\frac{i}{d} + k\right)$ for p large.

Fix an interval as above. $\forall c_1, c_2 \in I$ have a canonical Morita equivalence $\phi_{c_1,c_2} : A_{c_1} \sim A_{c_2}$.

Fix an irreducible module L over A_{c_0}, $c_0 \in I$.
Describe the function

$$d_L : I \longrightarrow \mathbb{Z}$$
$$c \longmapsto \dim \phi_{c_0,c}(L)$$

dimension of the corresponding module.

<u>Claim</u> d_L is a polynomial function
$D_L(c,p)$ (p fixed for now) total deg $2n-2$

$$D_L(c,p) = \chi\left(\mathcal{F}_{L,I} \otimes \mathcal{O}\left(\tfrac{c}{p}\right)\right)$$

Euler char. $K^0_{\mathbb{Q}}(\mathrm{Coh}_{X_n^0} X_n) \longrightarrow \mathbb{Q}$

$\mathcal{F}$ a certain virtual sheaf -- $\mathcal{O}(1)$ is uniquely defined in this way so can raise to any rational power after tensoring with $\mathbb{Q}$.

... coherent sheaf corresponding to an irreducible tensor the splitting bundle

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Crossing between intervals (present as conjecture, can almost prove).

1. The t-structure "doesn't depend on p", only on $\underline{I}$ (for p large), in particular lifts to characteristic 0.

(have tilting bundle over a finite localization of $\mathbb{Z}$ $\Rightarrow$ algebra whose specializations give our algebras for different p, indecomposable projectives specialize etc)

2. $D_L(c,p)$ is a polynomial, with coefficients independent of p.

$p^d D_L(c,p)$ has a nonzero limit as $p \to \infty$

$\overline{D_L}(\frac{c}{p})$

Suppose I_1, I_2 are consecutive intervals

c_0 = common boundary.

Get a function $\{\text{irreds for } A_c,\ c \in I_1\} \xrightarrow{m} \mathbb{Z}_+$

$m(L)$ = order of zero of $\overline{D_L}$ at c_0

$\iff$ dim of L as rep of A_{c_0-1} will be p^m

$\iff$ dim of support of corresponding module in category $\mathcal{O}$ for Cherednik algebras.

e.g. $n=3$: reps $\longleftrightarrow$ reps of S_3

$p \equiv 1 \bmod 3$ $\quad I: \frac{-(p+2)}{3} \quad \frac{p-1}{3}$

dims are $\quad \frac{(3c+2+p)(3c+1+p)}{6}$

$\frac{(3c-p+2)(3c-p+1)}{6}$

$\frac{2p^2-2-9c^2-9c}{6}$

Perverse equivalences

* The irreducibles for I_2 are <u>in bijection</u> with those for I_1 & characterized by fact:

$$L_i' \simeq L_i\,[m(L_i)] \bmod \langle L_j : m(L_j) < m(L_i) \rangle$$

--- ie filter category using partial ordering m & shift different pieces by correspondingly amounts.

Quantum cohomology

Okounkov–Pandharipande compute $QH^*(\mathrm{Hilb})$ equivariantly for $(\mathbb{C}^*)^2$

$\rightsquigarrow$ flat connection on $\mathbb{C}^*$ with fiber $H^*(\mathrm{Hilb})$. has regular singularities at $-\mu$: $\mu^d = 1$ $d \leq n$, $\mu \neq 1$

In fact have a family of connections

depending on equivariant parameters t_1, t_2.

Conjecturally the monodromy of this connection can be described in terms of bases attached to intervals & that the bases are vanishing cycles attached to singular points:

Singular points of the connection are $-\exp\left(2\pi i \frac{c}{p}\right)$ for c = endpoints of intervals & p large

Conjecture Bases corresponding to the intervals $\longleftrightarrow$ vanishing cycles of singular points, & monodromy $\longleftrightarrow$ autoequivalences of the derived category generated by T_c, $c \in Q_n$

Monodromy: have an autoequivalence T_c

$L_i \longmapsto L_i[2m(L_i)]$ mod smaller pieces

- analogous to long intertwiner for category $\mathcal{O}$

Bases for I obtained by "half-monodromy" of the connection