Roman Bezrukavnikov - Bridgeland Stability

Note Title 1/15/2008

[inspiration: M. Douglas, D. Joyce & R. Thomas on SLAGs, ...]

Physics motivation:

Branes in appropriate SUSY QFT give objects of a derived category $D(\mathrm{Coh}\,X)$ - but original theory depends on more parameters than $D(\mathrm{Coh}\,X)$ e.g. Kähler structure on X & B-field. — capture this dependence by <u>stability</u> structure: locally constant dependence of $D(\mathrm{Coh}\,X)$ on these "irrelevant" parameters so we get monodromy transformations.

Math motivation: • deep conjectural relations to quantum cohomology, Frobenius manifolds etc.
• provides tools to work out numerical invariants of representation categories

t-structures don't seem to have moduli but their enhancement to stability conditions do!

<u>t-structure</u> on a triangulated category:
way to record abelian category inside a triangulated one, like $\mathcal{A} \subset D(\mathcal{A})$

- give two subcategories $\mathcal{D}^{>0}$, $\mathcal{D}^{\leq 0}$ with $\mathrm{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{>0}) = 0$ & invariant under appropriate shifts, so that $\forall X \ \exists \ X^{\leq 0} \to X \to X^{>0}$ triangle.

Bounded t-structure: $\mathcal{D} = \bigcup_{a,b} \mathcal{D}^{\geq a} \cap \mathcal{D}^{<b}$

$\Rightarrow$ determined by heart $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$

Given $X \to Y \to Z$ triangle say Y is an extension of Z by X: "$Y \in X * Z$"

Octahedron $\Rightarrow$ $*$ is associative

$\mathcal{D} = \bigcup \mathcal{A}[a] * \mathcal{A}[a-1] \cdots * \mathcal{A}[a-b]$

& $\mathrm{Hom}(\mathcal{A}, \mathcal{A}[-d]) = 0 \quad d > 0$.

Def A slicing of a triangulated category $\mathcal{D}$ is a collection $P = P(\varphi)$ $\varphi \in \mathbb{R}$ satisfying

a. $P(\varphi + 1) = P(\varphi)[1]$

b. $\varphi_1 > \varphi_2 \Rightarrow \mathrm{Hom}(P(\varphi_1), P(\varphi_2)) = 0$

c. $\forall E \neq 0 \quad E \in P(\varphi_1) * P(\varphi_2) \cdots * P(\varphi_k)$
for some $\varphi_1 > \cdots > \varphi_k$: $E \in E_1 * \ldots * E_k$
[$\Rightarrow$ E_i defined uniquely by E]
$P(\varphi) =:$ semistable objects of phase φ

Ex $\quad \mathcal{D} = \mathcal{D}(\mathrm{Coh}\, X) \quad X$ projective curve
$\Rightarrow$ can use Harder - Narasimhan filtrations to give a slicing with the $P(\varphi)$'s given by {torsion sheaves} $[d]$ $\quad d \in \mathbb{Z}$
& {semistable bundles of given slope} $[d]$ $\quad d \in \mathbb{Z}$
[$\varphi \rightsquigarrow$ arctan of slope]

In general $P(\varphi)$ is abelian $\forall \varphi$, & its simple objects are called stable.

Def A stability condition on $\mathcal{D}$ is a pair (P, Z) with P a slicing & $Z: K^0(\mathcal{D}) \longrightarrow \mathbb{C}$ linear functional s.t. $Z(E) = m e^{i\varphi}$
$(m > 0) \quad \forall E \neq 0 \quad E \in P(\varphi)$

Example of D, P as above:

$Z = -\deg + i \operatorname{rank}$

<u>Notation</u> if $I \subset \mathbb{R}$, given a slicing P set
$P(I)$ = full subcategory of objects filtered by $P(\varphi)$ $\varphi \in I$.

Claim: A slicing determines a t-structure with
$D^{\geq 0} = P(\mathbb{R}_{>0})$, $\mathcal{A} = P((0,1])$

Claim: Can reconstruct a stability condition (P, Z) from Z & the t-structure:

given $Z, \mathcal{A}$ with $Z : K^0(D) = K^0(\mathcal{A}) \longrightarrow \mathbb{C}$

s.t. $Z[M] \in \mathbb{H}$ $\quad 0 \neq M \in \mathcal{A}$

(here $\mathbb{H}$ =  upper half plane + left $\mathbb{R}$-axis)

$\varphi(M) = \arg Z[M]$ $\quad 0 \neq M \in \mathcal{A}$

Say M is semistable if it has no subobjects of smaller phase $\rightsquigarrow P(\varphi)$.

Can characterize which pairs $\mathcal{A}, Z$ arise:

Claim A bounded t-structure & functional Z come from a stability condition iff

1. $Z: A \setminus \{0\} \longrightarrow \mathbb{H}$

2. Harder-Narasimhan property: every object in A has a filtration with semistable subquotients

Prop (Rudakov) HN property $\Longleftrightarrow$ chain conditions

No infinite increasing or decreasing chains with increasing/decreasing phases. (so automatic in finite length categories).

Let $\mathrm{Stab}(D) = \{(P,Z)\}$ set of stability conditions. Metric on $\mathrm{Stab}(D)$:

$m = |Z|$

$$d((P,Z),(Q,Z')) =$$

$$\sup_{0 \neq E \in D} \left\{ |\varphi^{min}_{\sigma_2}(E) - \varphi^{min}_{\sigma_1}(E)|, |\varphi^{top}_{\sigma_2}(E) - \varphi^{top}_{\sigma_1}(E)|, \left|\log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)}\right| \right\}$$

Bad example: D as before. Set

$Z = i(\deg + \tau \cdot \mathrm{rank})$: all along same ray $(\tau \in \mathbb{R})$

To exclude this define locally finite stability conditions:

$b - a \leq 1 \Rightarrow P((a,b))$ which is an exact category: (Z, P) is locally finite if $\exists\, \eta > 0$ s.t. $P(a-\eta, a+\eta)$ is finite length (locally finite) for $\forall a$

Now Let $\mathrm{Stab}(\mathcal{D})$ = locally finite stability conditions.

X algebraic variety $\Rightarrow$
$\mathrm{Stab}(X)$ = locally finite stability conditions on $D^b(X)$ s.t. Z factors through

$$\begin{tikzcd} K^0(X) \arrow[r, "ch"'] \arrow[dr, "Z"'] & H^{*}(X) \arrow[d, dotted] \\ & \mathbb{C} \end{tikzcd}$$

(numerical stability condition)

<u>Theorem</u> $\forall$ component $\Sigma \subset \mathrm{Stab}(X)$
$\exists\, V = V(\Sigma)$ vector subspace of $\mathrm{Hom}(H^{*}, \mathbb{C})$
s.t. $\pi : \Sigma \longrightarrow V(\Sigma)$ is a local homeomorphism

– ie locally if we vary Z can follow along the stability condition uniquely.

<u>Corollary</u> $Stab(X)$ is a complex manifold.

<u>Remark</u> $Gl_2\mathbb{R} \circlearrowright \mathbb{R}^2 = \mathbb{C}$, $\widetilde{Gl_2\mathbb{R}^+} \circlearrowright Stab X$

<u>Simple example</u> $\mathcal{A}$ Artinian & Noetherian abelian category (fin # irreducibles + finite length)

$\Rightarrow \mathbb{H}^{\#irred} \subset Stab(\mathcal{D}(\mathcal{A}))$

given by Z of each irreducible.

When charge of an irreducible hits the boundary of $\mathbb{H}$ should <u>tilt</u> the t-structure along corresponding subcategory

Given an abelian category $\mathcal{A}$ & subcategories $\mathcal{T}, \mathcal{F}$

[ex $\mathcal{A} = Coh\, X$ $\mathcal{T}$ = torsion, $\mathcal{F}$ = torsion-free]

with $Hom(\mathcal{T}, \mathcal{F}) = 0$ & $\mathcal{T} * \mathcal{F} = \mathcal{A}$

$\Rightarrow$ change t-structure by shifting on $\mathcal{T}$:

define a new $\mathcal{A}' \subset \mathcal{D}(\mathcal{A})$:

$$C \in \mathcal{A}' \iff H^0(C) \in \mathcal{T},\ H^{-1}(C) \in \mathcal{F}$$
$$H^i(C) = 0 \quad i \neq 0, -1$$

After crossing this wall where $Z(L_i) \in \mathbb{R}_+$
new irreducibles will be $L_i[1]$
& L_j' with $L_j' = L_j$ modulo $\langle L_i \rangle$.

(take universal extension of L_j by L_i to kill Ext between them:
$$0 \to \mathrm{Ext}^1(L_j, L_i) \otimes L_i \longrightarrow L_j' \to L_j \to 0 \quad)$$

<u>Examples</u> $\mathcal{D} = D(X)$ X algebraic variety
- worked out for X smooth projective curve, K3 surface, or $\mathcal{D} = D_Z(X) \subset D(X)$
sheaves supported on $Z \subset X$ for X some local Calabi-Yaus : $X = \mathrm{Tot}\, K_Y$ Y Fano surface
(Z = zero section) or
X = resolved ADE singularity $\supset Z$ = central fiber
(in these cases a component of Stab is calculated)

Form of the answer:

$V = (K^0_{\mathbb{C}})^* \supset U$ open (cone ∖ hyperplanes)

[or really image in cohomology]

$\tilde{U} \longrightarrow U$ covering is then identified as a component of Stab X & the group of deck transformations of $\tilde{U}/U$ acts by automorphisms on $D(X)$ ~ e.g. actions of braid group arise.

This structure matches the expectations from physics for dependence on auxiliary parameters.

Technology: construct by hand an initial t-structure ... never tautological one, usually has Artinian heart ...

eg for $K_{\mathbb{P}^2} = \mathrm{Tot}\ \mathcal{O}_{\mathbb{P}^2}(-3)$: resolution of $\mathbb{C}^3/\mathbb{Z}_3$. Known $D(K_{\mathbb{P}^2}) \simeq D_{\mathbb{Z}_3}(\mathbb{C}^3)$

... transport the orbifold t-structure on $D_{\mathbb{Z}_3}(\mathbb{C}^3)$ (supported at 0)

For $X = \widetilde{\mathbb{C}^2/\Gamma}$ again $D(X) \simeq D_\Gamma(\mathbb{C}^2)$, transport tautological t-structure.

$\Rightarrow$ Get $\mathbb{H}^n \subset \mathrm{Stab}$. Now want to extend further $\Rightarrow$ build 'automorphisms': reflections in walls that tell us covering transformations: images of $\mathbb{H}^n$ cover a component in Stab.

For K3, $K_{\mathbb{P}^2}$, $\widetilde{\mathbb{C}^2/\Gamma}$ etc reflections are all generated by spherical objects.

Def $S \in D$ is spherical of dimension n if
$$\mathrm{Ext}(S,S) \simeq H^*(S^n)$$

Reflection in S: $M \longmapsto \mathrm{Cone}\left(\mathrm{Hom}(S,M) \otimes S \longrightarrow M\right)$

- For $\widetilde{\mathbb{C}^2/\Gamma}$ get covering of
$$\left(\mathfrak{h}^*_{\mathrm{aff}} = \mathfrak{h}^*_{\mathrm{fin}} \otimes \mathbb{C}\right) \setminus H_{\alpha,n} \quad \text{affine coroot hyperplanes}$$

- K3 $\Rightarrow$ N = Mukai lattice, set
$$N \otimes \mathbb{C} \supset P \supset P^+ \text{ component}$$

find $\widetilde{P^+ \sim \cup S_\delta}$ $\qquad \delta \in N$, $(\delta, \delta) = \pm 2$

Generalization of ADE example:

(Bezrukavnikov, Arno) $\qquad$ $T^* G/B \xrightarrow{\mu} \mathfrak{g} \supset$ nilpotent cone

$$\begin{array}{ccc} T^*G/B & \xrightarrow{\mu} & \mathfrak{g} \\ \cup & & \cup \\ X & \xrightarrow{\mu} & S \end{array}$$

$S =$ transversal slice to nilp orbit

$D = D_Z(X)$

$Z =$ central fiber

$\mathrm{Stab}\, D \supset \widetilde{\text{half} \sim H_{\sigma, n}}$.

Initial t-structure here is the exotic one.