R. Bezrukavnikov - Stability Conditions 2

Note Title 1/22/2008

1. Bridgeland's calculation of stability conditions $\mathrm{Stab}(\widetilde{\mathbb{C}^2/\Gamma})$ on the minimal resolution of a Kleinian singularity

D: triangulated category

$D \ni M$ is spherical if $\mathrm{Ext}^\bullet(M,M) \cong H^\bullet(S^d)$

M is exceptional if $\mathrm{Ext}^\bullet(M,M) = \mathbb{C}$

If $D = D(X)$ X Calabi-Yau of dim d & M spherical $\Rightarrow$ its dimension is $d = \dim X$

If $Z \overset{i}{\subset} X$ smooth divisor & X CY, $M \in D^b(Z)$ is exceptional $\Rightarrow i_* M$ is spherical:

$$\mathrm{Ext}(i_* M, i_* M) = \mathrm{Ext}(i^* i_* M, M)$$

$i^* i_* M \to M$, $M[1] \otimes K_Z^{-1} \to i^* i_* M$ from CYness of X

since $\mathrm{Ext}^i(M, K_Z \otimes M) = \begin{cases} \mathbb{C} & i = d-1 \\ 0 & i \neq d-1 \end{cases}$

Theorem (Bridgeland) $\mathcal{D} \subset D^b Coh(\widehat{\mathbb{C}^2/\Gamma})$
[sheaves with support on central fiber of
$\pi : \widehat{\mathbb{C}^2/\Gamma} \longrightarrow \mathbb{C}^2/\Gamma$]
$\Rightarrow$ there is a component of Stab $\mathcal{D}$ which is
a covering space Σ of $\mathfrak{h}^*_{aff} \setminus H_{\alpha,n}$
affine Cartan of type ADE $\longleftrightarrow \Gamma$ minus
affine coroot hyperplanes

- The affine braid group acts on $\mathcal{D}$, preserving Σ & acting by deck transformations over $\mathfrak{h}^*_{aff}/W_{aff}$
[in fact $\mathbb{Z} \times B_{aff}$ acts]

For S a spherical object $\Rightarrow R_S$ autoequivalence of the derived category $Hom(S,F) \otimes S \to F \longrightarrow R_S(F)$

Observation (Seidel-Thomas): $60°$ angle between objects $\Rightarrow$ braid relation: i.e
$Ext(S_2, S_1) = \mathbb{C} = Ext(S_1, S_2) \Rightarrow$
$R_{S_1} R_{S_2} R_{S_1} = R_{S_2} R_{S_1} R_{S_2}$

So the braid group action on our D is easy to produce via spherical objects!

$\widehat{\mathbb{C}^2/\Gamma}$ is CY & have divisors $\widehat{\mathbb{C}^2/\Gamma}_0 = \bigcup \mathbb{P}^1_\alpha$

$\mathbb{P}^1_\alpha \xhookrightarrow{i_\alpha} \widehat{\mathbb{C}^2/\Gamma} \Rightarrow i_{\alpha *}(\mathcal{O})$ is spherical

& they satisfy braid relations $\Rightarrow$ gives <u>finite</u> braid group action on D

Collection $i_{\alpha *}\mathcal{O}(-1)[1]$, $\mathcal{O}(\widehat{\mathbb{C}^2/\Gamma})_0$ give spherical collection generating full B_{aff} action.

Under McKay correspondence
$$D^b(\mathrm{Coh}\ \widehat{\mathbb{C}^2/\Gamma}) \simeq D^b(\mathrm{Coh}^\Gamma \mathbb{C}^2$$

these spherical objects correspond to

$\rho \otimes \mathbb{C}_0 \qquad \rho \in \mathrm{Irrep}(\Gamma)$

(equivariant skyscrapers at 0)

These spherical functors relate the standard t-structure to the t-structure obtained by tilting the spherical object

$$\mathrm{Tilt}_{S_i}\, \mathrm{Coh}^{\Gamma}\mathbb{C}^2 \Leftrightarrow R_{S_i}\, \mathrm{Coh}^{\Gamma}\mathbb{C}^2$$

To describe our component of $\mathrm{Stab}\, D$:
the component Σ will be the one containing all pairs $\{(A, Z) : A = \mathrm{Coh}^{\Gamma}\mathbb{C}^2\} \cong \mathbb{H}^n$

$\Sigma = \bigcup_{b \in B_{\mathrm{aff}}} b \cdot (\mathbb{H}^n)$: observation is that Σ is a component!

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2. $D = D^b\, \mathrm{Coh}_{\mathbb{P}^2}(K_{\mathbb{P}^2})$

Reorem-Construction (Bridgeland)

$\mathrm{Stab}\, D \supset$ explicit open subset $M \circlearrowleft B$
affine braid group of type A_2 $= B_{\mathrm{aff}}(A_2)$

– conjecturally related to quantum cohomology $QH^*(\mathbb{P}^2)$, to be discussed.

In this situation B doesn't act on D
but will act on a collection of t-structures

$$(t,Z) \in \mathcal{M} \longrightarrow H^*(\mathbb{P}^2) \text{ is not (expected to be)}$$

s/ a covering of its image.

$$Z \in \mathrm{Hom}(K^0, \mathbb{C})$$

Our category is still generated
by spherical objects, but they don't satisfy
$60°$ relation $\longrightarrow$ probably generate free group
of automorphisms.

$$K_{\mathbb{P}^2} \longrightarrow \mathbb{C}^3/\mathbb{Z}_3 \hookleftarrow \text{acting by scalars}$$

|| crepant resolution

$$D^b \mathrm{Coh}^{\mathbb{Z}/3}(\mathbb{C}^3)$$

[In general expect crepant resolution derived equivalent to an NC resolution]

Let $1, \xi, \xi^2$ be the characters of $\mathbb{Z}/3$
as skyscraper sheaves at 0
Exts between the corresponding skyscrapers are
$\mathbb{Z}/3$ isotypic components of exterior algebra :

$$\mathrm{Ext}(\xi^i, \xi^j) = \left(\xi^{j-i-}\otimes \Lambda^1\mathbb{C}^3\right)^{\mathbb{Z}_3} \simeq \mathbb{C}^3 \quad i\neq j$$

(either Λ^1 or Λ^2)

& ξ^i are spherical.

$$M \to \{(A,Z) : A = \mathrm{Coh}^{\mathbb{Z}_3}(\mathbb{C}^3)\} \simeq \mathbb{H}^3$$

$B \longleftrightarrow B$ action on solutions to Markov's equation

$$a^2+b^2+c^2 = abc \qquad a,b,c\in\mathbb{Z}$$

... form a tree with edges $\longleftrightarrow$ elementary operations! fix b,c get two solutions a_0, a_1 so can replace a_0 by a_1

Relates to work of Rudakov seminar on exceptional collections on $\mathbb{P}^2$

An exceptional collection in a triangulated category D is $E_1,\dots,E_n$ all exceptional

& $\mathrm{Ext}^\bullet(E_j, E_i)=0 \quad j>i$

Strong exceptional collection:

$$\mathrm{Ext}^s(E_i, E_j) = 0 \quad s\neq 0 \quad \forall i,j$$

Full exceptional collection : generates D.

Theorem (Rudakov, Gorodentsev, Bondal ?)
The braid group B_n acts on the set of exceptional collections

[Topology examples : $E_i = j_{i*} \mathbb{C}$ or $j_{i!} \mathbb{C}$ for $j_i : S_i \subset X$ stratification of topological space with contractible strata]

Exceptional collection $\Rightarrow$ embedding functor $\langle E_i, E_j \rangle \subset D$ has adjoint (admissible subcategory) $\rightsquigarrow$ orthogonalization.

e.g. Left mutation $\quad L_E F \to \operatorname{Hom}(E,F) \otimes E \xrightarrow{ev} F$

elementary switches $\quad \ldots\ldots E_{i-1} E_i \ldots$

$$\Rightarrow \cdots\cdots E_i \, L_{E_i}(E_{i-1}) \cdots\cdots$$

This operation doesn't usually preserve <u>strong</u> collections ...

A **helix** of period n is a sequence of exceptional objects E_i $i \in \mathbb{Z}$ s.t. for all i,
$\underbrace{E_i, \ldots, E_{i+n}}$ is an exceptional collection
& E_{i+n+1} is obtained from E_i by the long word in the braid group.

$E_1 \ldots E_n$ is **simple** if $Ext^{\neq 0}(E_i, E_j) = 0$ $\forall i \in \mathbb{Z}$ in the helix

... very rare!

Assume Rank $K^0(X) = \dim X + 1$.
e.g. $X = \mathbb{P}^n$, odd dimensional quadrics,
a few more. $\Rightarrow$
$E_1, \ldots E_n$ strong full exceptional collection is simple (Bondal - Polishchuk)

Serre functor $S: F \mapsto F \otimes \omega[\,]$
$\Rightarrow E_{i+n+1} = S(E_i)$.

Such collections are invariant under mutation

$X = \mathbb{P}^2$: can apply elements of B_{aff} to generate strong exceptional collections from a given one $\mathcal{O}(-2)$ $\mathcal{O}(-1)$ $\mathcal{O}$

a, b, c = dimension of Homs between these objects $\Longrightarrow$ solutions of Markov equation

Any strong exceptional collection defines a t-structure $D \simeq D(\mathrm{End}(\oplus E_i)\text{-mod})$

Simples are $F_{n-j}[j]$ where (F) is the strong exceptional collection dual to ours: given by mutation by the long element in the permutation group.

mutation of exceptional sets $\longleftrightarrow$ tilting of t-structures

Such a t-structure defines a unique t-structure on $D \subset D(\mathrm{Coh}\, K\mathbb{P}^2)$ st i_* is exact, with same irreducibles

$\Rightarrow$ will get t-structures with 3 irreducibles which are spherical, with a cyclic order arrow $\leadsto$ nonzero Ext^1.

Dubrovin's conjecture

Def A Frobenius manifold is a Riemann manifold (M, g) with $\circ$ a commutative multiplication on the tangent sheaf which is flat
$g(X \circ Y, Z) = g(X, Y \circ Z)$ s.t. $\exists$ potential φ with $g(X \circ Y, Z) = XYZ(\varphi)$

+ Euler vector field E, $\mathrm{Lie}_E(\circ) = \circ$
$\mathrm{Lie}_E(g) = (2-d)g$ $d =$ charge
+ identity

GW theory $\leadsto$ Frobenius manifold structure on formal neighborhood of $0 \in H^*(M)$, sometimes can analytically continued

Def A Frobenius manifold is <u>tame</u> if
$u : x \longmapsto E \cdot x$ is regular semisimple

$\Longrightarrow$ nice local theory: eigenvalues of u
define local coordinates $M \longrightarrow \mathbb{C}^n - \Delta / S_n$

$E = \sum u_i \partial_{u_i}$ $\qquad \partial_{u_i} \circ \partial_{u_j} = \delta_{ij} \partial_{u_i}$

Semisimple (or tame) Frobenius manifolds arise
as spaces of isomonodromic deformations of
connections on the line

More precisely $M \times \mathbb{C}$ carries canonical 1-parameter
family of flat connections - in fact two such
related by Fourier - Laplace transform
2nd structured connection: regular singularities
at $u_i \in \mathbb{C}$
1st has irregular singularity at ∞, regular at 0

Conjecture of Dubrovin: The Frobenius manifold associated to $QH^*(X)$ is tame iff $D(X)$ is generated by an exceptional collection ($\Rightarrow X$ Fano)

& The matrix $E = \chi\left(\mathrm{Ext}(E_i, E_j)\right)$ of Euler characteristics is the Stokes matrix of the first structural connection

In the case $X = \mathbb{P}^2$ the corresponding Frobenius manifold is $M = \widetilde{\mathrm{Conf}_3(\mathbb{C}) \setminus Z}$ universal cover

Z = configurations containing 0.

$\rightsquigarrow$ relate to stability manifold!

Conjecture: M = our previous $M \subset \mathrm{Stab}\, K_{\mathbb{P}^2}$.

Claim: A basis of flat sections of the second structural connection around 0 identifies with the basis of irreducible objects in the t-structure constructed above.

$\mathcal{B}$ acts by monodromies on flat sections & on exceptional collections, & above identification will identify them.

Key point: have an ODE whose monodromy controls numerics of tiltings / wall crossings in the derived category.

More generally: Toledano-Laredo & Bridgeland, Kontsevich & Soibelman: use Donaldson-Thomas invariants to construct such a formalism in a more general situation...

<u>Problem</u> Bridgeland et al give wall crossing formulas. Kazhdan-Lusztig formalism gives other such formulas. Bring them under one roof!

Usually our category has a grading (spaces have $\mathbb{C}^*$ action) $\rightsquigarrow$

K^0 has a natural one parametric deformation

$$K^0_q = K^0(\mathrm{Coh}^{\mathbb{C}^*}) \circlearrowleft q : \mathcal{F} \rightsquigarrow \mathcal{F}(1)$$

Often have an interesting t-structure $\mathcal{A}$
$\Rightarrow \mathbb{H}^n \subset \mathrm{Stab}\, D$, want to understand behavior of $\mathcal{A}$ under "wall crossing" in Stab

$$\mathbb{H}^n \supset \mathbb{R}^n$$

Bridgeland: if we have polynomial vanishing of order n along wall, should apply tilting along the boundary involving shift by n

Kazhdan-Lusztig: K^0_q has a $\mathbb{C}[q, q^{-1}]$-valued pairing & our bases are asymptotically orthonormal (i.e. regular at $q=0$, give ortho bases at $q=0$)

irreds on which our functional vanishes to order k: multiply by q^k & then apply Gram-Schmidt orthonormalization.