L. Rybnikov - Opers & Shift of Argument Subalgebras

Note Title 4/1/2008

(w/ B. Feigin & E. Frenkel)

$\mathfrak{g}$ s/s Lie algebra$/\mathbb{C}$, $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$
with bracket $\{x,y\} = [x,y]$

Poisson center $Z S(\mathfrak{g}) = S(\mathfrak{g})^{\mathfrak{g}} \simeq \mathbb{C}[\varphi_1, \dots \varphi_\ell]$
$\ell = \mathrm{rk}\, \mathfrak{g}$

$\mu \in \mathfrak{g}^* \rightsquigarrow$ first order diffop (derivation)
$\partial_\mu : S(\mathfrak{g}) \to S(\mathfrak{g})$

$A_\mu := \mathbb{C}\left[\partial_\mu^k \varphi_i\right]_{i=1,\dots \ell,\ k=0,1,\dots \deg \varphi_i - 1}$
shift of argument subalgebra

<u>Theorem</u> (Fomenko - Mishchenko)

1. A_μ is Poisson commutative
2. $\mu \in \mathfrak{g}^*_{reg}$: $\mathrm{trdeg}\, A_\mu = \frac{1}{2}(\dim \mathfrak{g} + \mathrm{rk}\, \mathfrak{g})$
$= \dim \mathfrak{b}$

— maximal possible transcendence degree

[in fact for $\mu \in \mathfrak{g}^*_{reg}$ this is a <u>maximal</u> commutative subalgebra]

<u>Theorem</u>(R) 1. $\exists\, \mathcal{A}_\mu \subset U\mathfrak{g}$ commutative subalgebra with $\operatorname{gr} \mathcal{A}_\mu = A_\mu$

2. $\mathcal{A}_\mu$ is unique for generic μ.

<u>Problem</u> For M a $\mathfrak{g}$-module, compute the spectrum of $\mathcal{A}_\mu$ on M

... natural to expect it to be simple for simple M.

Two opposite cases:

1. $\mu \in \mathfrak{h}_{reg} \Rightarrow$ generators $\mathcal{A}_\mu^{(1)} = \mathfrak{h}$,

$$\mathcal{A}_\mu^{(2)} = S^2\mathfrak{h} + \left\{ \sum_{\alpha > 0} \frac{\langle h, \alpha \rangle}{\langle \mu, \alpha \rangle} e_\alpha f_\alpha : h \in \mathfrak{h} \right\}$$

... for generic μ, $\mathcal{A}_\mu =$ centralizer (")

2. $\mu = f$ principal nilpotent :

linear generators $\mathcal{A}_f^{(1)} = Z_{\mathfrak{g}}(f)$
+ quadratic, cubic
additional grading: $\deg \partial_f^k \varphi_i = k$
$\Longleftrightarrow$ principal gradation on $\mathfrak{b}$

Theorem 1) Finite dim irred modules are cyclic over $\mathcal{A}_f$,
$V_\lambda = \mathcal{A}_f \cdot v_\lambda$ & the annihilator $\mathrm{Ann}_{\mathcal{A}_f}(V_\lambda)$
is generated by a regular sequence $P_\alpha \in \mathcal{A}_f$
(α: pos roots) with $\deg P_\alpha = \langle \lambda+\rho, \alpha^\vee \rangle$
[+ kernel in $Z(U_{\mathfrak{g}})$]

2) $V_\lambda \ni$ cyclic vector for each $\mu \in \mathfrak{g}^*_{\mathrm{reg}}$
+ for generic μ have a simple spectrum

$\hat{\mathfrak{g}}$ = affine Lie algebra corresponding to $\mathfrak{g}$
$= \mathfrak{g}((t)) + \mathbb{C}K = \mathfrak{g}[[t]] + \mathbb{C}K + t^{-1}\mathfrak{g}[t^{-1}]$

$$\underbrace{\mathfrak{g}[[t]] + \mathbb{C}K}_{\hat{\mathfrak{g}}_+} \qquad \underbrace{t^{-1}\mathfrak{g}[t^{-1}]}_{\hat{\mathfrak{g}}_-}$$

<u>Feigin-Frenkel</u> $\quad \tilde{U}(\hat{\mathfrak{g}})_{-h^\vee} \supset Z(\hat{\mathfrak{g}}) \cong \mathbb{C}[\tilde{\varphi}_i^{(k)}]$

completed critical level enveloping algebra $\qquad$ large center

where $\deg_{PBW} \tilde{\varphi}_i^{(k)} = \deg \varphi_i$

$\deg_{\text{loop rotation}} \tilde{\varphi}_i^{(k)} = k$

Take hamiltonian reduction

$$Z(\hat{\mathfrak{g}}) \to \Big[U\mathfrak{g} \otimes \tilde{U}(\hat{\mathfrak{g}})_{-h^\vee} \Big/ U\mathfrak{g} \otimes \tilde{U} \cdot \underbrace{\hat{\mathfrak{g}}_+^{diag}}_{\text{diagonal embedding in the tensor product}} \Big]^{\hat{\mathfrak{g}}_+}$$

$$\begin{array}{c} \big\downarrow \wr \\ U\mathfrak{g} \otimes U\hat{\mathfrak{g}}_- \end{array}$$

$\mu \in \mathfrak{g}^* \rightsquigarrow$ character $\hat{\mathfrak{g}}_- \to \mathbb{C}$

$$xt^{-1} \longmapsto \langle \mu, x \rangle$$
$$xt^k \longmapsto 0 \qquad k \neq 1$$

- composing with this gives

$$Z(\hat{\mathfrak{g}}) \longrightarrow U\mathfrak{g} \otimes U\hat{\mathfrak{g}}_- \longrightarrow U\mathfrak{g} : \text{image is } A_\mu$$

$$Z(\hat{\mathfrak{g}}) = \mathrm{Fun}(\mathrm{Op}_{{}^L\mathfrak{g}}(D^\times))$$

Opers $\mathrm{Op}_{{}^L\mathfrak{g}}(D^\times) = \{\partial + p_{-1} + v(t) : v(t) \in {}^L\mathfrak{b}_+((t))\} / {}^L N((t))$

<u>Theorem</u> (Drinfeld-Sokolov)

$$\mathrm{Op}_{{}^L\mathfrak{g}}(D^\times) \simeq \{\partial + p_{-1} + v(t)\}$$

$v(t) \in Z_{{}^L\mathfrak{g}}(p_1)((t))$ where p_{-1}, ρ, p_1 are a principal $\mathfrak{sl}_2$ triple

$$\begin{array}{ccccc} \mathrm{Fun}(\mathrm{Op}_{{}^L\mathfrak{g}}(D^\times)) & \longrightarrow & \mathrm{Fun}(\mathrm{Op}^{RS}_{{}^L\mathfrak{g}}(D)) & \longrightarrow & \mathrm{Fun}(\widetilde{\mathrm{Op}}) \\ \| & & \| & & \\ Z(\hat{\mathfrak{g}}) & & \text{Hamiltonian reduction} & & A_\mu \end{array}$$

$\widetilde{\mathrm{Op}} = \mathrm{Op}_{{}^L\mathfrak{g}}(\mathbb{P}^1, 0, \infty)$: regular sing at 0

2nd order pole at ∞

2-residue at ∞ is $\mu \in {}^L\mathfrak{g}/{}^LG$.

Let $p_1, \dots p_\ell$ be a basis of $z_{\mathfrak{g}}(p_1)$

Write oper as $\partial + p_{-1} + \sum_i^\ell \sum a_j^i t^j p_i$

To get spectrum acting on a rep V_λ:
fix residue at 0 to be $-\lambda-\rho$

Frenkel - Gaitsgory: support of <u>fin. dim</u>
rep $\subset$ opers with trivial monodromy at 0

<u>Key lemma</u> There is no monodromy-free oper
with irregular sing. of order 2 at ∞
& 2-residue $= 0 \in \mathfrak{g}^L/G^L$

The Poincaré polynomial of functions on such opers
is precisely the same as the graded character
of V_λ ... so now use that A_f contains
the centralizer of f, $f^k v_\lambda = v_{w_0(\lambda)}$

$\rightsquigarrow$ find $\mathrm{Im}(A_f \to \mathrm{End}\, V_\lambda)$
$=$ Functions on monodromy-free opers!

Now use a deformation argument to show the same holds for A_μ ...

.... V_λ will no longer be cyclic, but there'll be another cyclic vector

// Identifying $V_\lambda = A_f / \mathrm{Ann}\, V_\lambda$ get structure of a nilpotent Frobenius ring on V_λ !

Also $V_\lambda \cong H^*(Gr_\lambda)$ & note

$$A_f > U(Z(f)) = H^*(Gr) \curvearrowright V_\lambda$$

– hope to get a topological description of full ring structure.

GL_n .. deformation of Gelfand-Tsetlin ..