Note Title 4/3/2008

O. Schiffman – Finite dimensional representations of DAHA (d'après Varagnolo-Vasserot)

I. Classification of (spherical) fin dim reps of DAHA
(known before in type A : Berest-Etingof-Ginzburg)

II. (Conjectural) "L-packet phenomenon"
combinatorics for DAHA fits combinatorics (Reeder)
of p-adic group representations (tamely ramified)

Plan I. – algebraic description of DAHA
- geometric construction of $\mathrm{Irr}(\mathcal{O}_h)$
using K-theory of affine Springer fibers
- sketch of proof of main theorem

II – statement

I. DAHA etc

G_0 connected & simply connected
reductive / $\mathbb{C}$ $G_0 \supset B_0 \supset T_0$

Root datum $(X_0, \Delta_0, Y_0, \Delta_0^\vee)$

$X_0 = \mathrm{Hom}(T_0, \mathbb{C}^*)$ $Y_0 = \mathrm{Hom}(\mathbb{C}^*, T_0)$

Δ_0 – root system $\Delta_0^\vee$ – coroot system

Π_0 – simple roots

W_0 - Weyl group of G_0 $\Rightarrow$ $\{s_\alpha, \alpha \in \Pi_0\}$

$\Delta = \underbrace{\Delta_0 \times \mathbb{Z}\delta}_{\Delta^{re} \text{ real roots}} \cup \underbrace{\{0\} \times \mathbb{Z}\delta}_{\Delta^{im} \text{ imaginary}}$

$W = W_0 \ltimes Y_0$ acts on $X_0 \times \mathbb{Z}\delta$

<u>Def</u> $\mathbb{C}_{q,t} = \mathbb{C}[q^{\pm 1}, t^{\pm 1}]$ q: modular parameter

Double Affine Hecke Algebra $\underline{H}$:

generators $\{t_w, w \in W\} \cup \{x_\lambda, \lambda \in X_0 \oplus \mathbb{Z}\delta\}$

relations : $\cdot$ $\{t_w\}$ generate affine Hecke algebra of W

- $x_\delta = q$ $\quad x_\lambda x_\mu = x_{\lambda+\mu}$

- $x_\lambda t_{s_\alpha} - t_{s_\alpha} x_{\lambda - \langle\alpha^\vee, \lambda\rangle\alpha} =$

$$(t - t^{-1}) x_\lambda (1 + x_{-\alpha} + \dots + x_{-\alpha}^{\langle\lambda,\alpha^\vee\rangle - 1})$$

$\alpha \in \Pi = \Pi_0 \cup \{0\}$ $\quad$ if $\langle\lambda, \alpha^\vee\rangle \geq 0$

- contains two affine Hecke algebras

$H_X = \langle t_w, w \in W_0 \ \& \ x_\lambda, \lambda \in X_0 \rangle$

& $H_Y = \langle t_w,\ w \in W \rangle$

- for G_0 & $G_0^\vee$.

<u>Notation for torus</u>: T_0 = torus for G_0

$T_0 \times \mathbb{C}^*_\delta = T_d$, $X^*(T_d) = X_0 \oplus \mathbb{Z}\delta$

$H = T_d \times \mathbb{C}^*$: an element of H is

(s, τ, ξ) gives parameters.

$s \in T_0$, $\tau \in \mathbb{C}^*_\delta$, $\xi \in \mathbb{C}^*$,

<u>Weights</u> If M is an $\underline{H}$-module & $h \in H$

$\Rightarrow M_h = \{ z \in M : \forall \lambda,\ (X_\lambda - \lambda(h))^r \cdot z = 0 \quad r \gg 0 \}$

<u>Category $\mathcal{O}$</u>:

$\mathcal{O}_h(\underline{H}) = \{ M \mid$ fin. gen., $\mathbb{C}[X_0]$-locally finite

$h = (s, \tau, \xi)$ $\quad q \mapsto \tau^{-1}$, $t \mapsto \xi$ act as scalars, & $M_{h'} = 0$ if $h' \notin W \cdot h \}$

$\mathcal{O}(\underline{H}) = \bigoplus_h \mathcal{O}_h(\underline{H})$ block decomposition

& each block $\mathcal{O}_h(\underline{H})$ is a finite length category.

<u>Problem</u> : Classify $\mathrm{Irr}(\mathcal{O}_h(\underline{H}))$ (✓)
& finite dimensional irreducibles (?)

From now on: fix τ, ξ not roots of unity
& assume $\xi = \tau^{c/2}$ with c rational $c = \frac{k}{m}$
($m > 0$, k, m rel. prime) --- this
number $c = \frac{k}{m}$ controls the representation theory!

<u>Polynomial reps</u> $\mathbb{C}$ = "trivial" rep of H_X
$(x_\lambda \longmapsto t^{2(\lambda, \rho^\vee)}$ $t_{s_i} \longmapsto t.)$

$$P_Y = \underline{H} \underset{H_X}{\otimes} \mathbb{C} \cong \mathbb{C}_{q,t}[Y_0]$$

$$P_X = \underline{H} \underset{H_Y}{\otimes} \mathbb{C} \cong \mathbb{C}_{q,t}[X_0]$$

Def An $\underline{H}$-module M is spherical if $P_Y \twoheadrightarrow M$ (Y-spherical if $P_X \twoheadrightarrow M$)

Specialize: $q \mapsto \tau^{-1}$ $\quad t \mapsto \xi = \tau^{c/2}$

$\Rightarrow P_{Y,c} \in \mathcal{O}_{h_c}(\underline{H}_c)$ ($P_{X,c}$ <u>not</u> in category $\mathcal{O}$)

<u>Problem</u>: Classify spherical $\underline{H}_c$-modules which are finite dimensional

--- not achieved geometrically...

What is achieved in this paper: geometric classification of spherical $\underline{H}_c$-modules which come from the rational Cherednik algebra (rational DAHA)

-- nilpotent reps of rational DAHA
$\longleftrightarrow$
unipotent reps of degenerate DAHA

all irreps of degenerate DAHA
$\longleftrightarrow$
all irreps of DAHA

H' degenerate DAHA: are affine Hecke $\rightsquigarrow$ group algebra of W

H'' rational DAHA [$\cong$ $\mathbb{C}[W_0] \ltimes \mathbb{C}[X] \ltimes \mathbb{C}[Y_0]$]

One defines $\mathcal{O}_h$, spherical etc for H', H''

<u>Equivalences</u> $\mathcal{O}^{nil}(H'') \xrightarrow[\text{exponential}]{\simeq} \mathcal{O}^{uni}(H')$

$\mathcal{O}(H') \simeq \mathcal{O}(\underline{H})$ (like Lusztig for AHA)

<u>Theorem</u> (Guay-Ginzburg-Opdam-Rouquier, Dunkl-Opdam, Berest-Etingof-Ginzburg, Dunkl)

Fix c as above $= \frac{k}{m}$

i) P''_c has a unique simple quotient L''_c

ii) $L''_c = P''_c$ if $k>0$

iii) L''_c is also a quotient of P''_X

[have involution switching X & Y and fixing simple reps]

So restricting to reps coming from H'' $\Longleftrightarrow$ restricting to reps which are quotients of both X & Y polynomial reps.

Corollary Fix $c \Rightarrow \exists$ at most one f.d spherical $\underline{H}_c$-rep coming from H_c^{rr}.
If it exists, $L_c \twoheadleftarrow P_\chi$ quotient of P_χ.

[spherical .... gen by 1-dim finite Hecke rep. ...]

<u>Problem</u> Determine when is this L_c finite dimensional?

<u>Statement of main theorem</u> $c = \frac{k}{m}$

<u>Theorem</u> L_c is finite dimensional iff $\begin{cases} k < 0 \\ c \text{ is an } \underline{\text{elliptic number}} \text{ of } W_0 . \end{cases}$

<u>Def</u> $\rho : W_0 \to \mathrm{End}\, \mathfrak{h}_0$ reflection representation
$w \in W_0$ is <u>regular</u> if it has a regular eigenvector, & <u>elliptic</u> if $1 \notin \mathrm{spec}(w)$
$\Leftrightarrow$ $w \notin$ any proper parabolic subgroup

A regular number is the order of a regular element
An elliptic " " " " " " elliptic "

e.g. Coxeter element is elliptic $\Rightarrow$
Coxeter # is always an elliptic #.
Type A: this is the only elliptic number.

<u>Sketch of proof</u>

- Geometric construction of all $\mathrm{Irr}(\mathcal{O}_h(H_c))$
- $P_{X,c}$ is a "standard" module with a simple top
- Determine when this top is finite dim.

II. Geometry & Springer fibers

$G = G_0((\varepsilon))$, $\hat{G}$ = Kac-Moody group

$1 \longrightarrow \mathbb{C}^\times_{\omega_0} \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1$

$\tilde{G} = \hat{G} \rtimes \mathbb{C}^\times_\delta$ loop rotation

(Kazhdan-Lusztig result)
Nilpotent cone: $\mathcal{N} = \{$top. nilpotent elts in $\tilde{\mathfrak{g}}\}$
$= \{x \in \tilde{\mathfrak{g}} : \mathrm{ad}\,x^N \longrightarrow 0$ in adic topology$\}$

Fix $\mathfrak{b}_0$, $\mathfrak{b}$ = Iwahori $= \mathfrak{b}_0 \oplus \varepsilon\mathfrak{g}[[\varepsilon]]$

$\mathfrak{b} \supset \mathcal{U} = \mathfrak{n}_0 \oplus \varepsilon\, \mathfrak{g}[[\varepsilon]]$ radical

$\Rightarrow \mathcal{N} = \mathrm{Ad}\, \widehat{G} \cdot \mathcal{U}$

e.g. $\varepsilon \mathfrak{h}[[\varepsilon]] \subset \mathcal{N}$, $\mathcal{N}_0[\varepsilon, \varepsilon^{-1}] \subset \mathcal{N}$.

An element $x \in \widehat{\mathfrak{g}}$ is regular semisimple if its centralizer $\widehat{G}_x$ is a maximal torus over $\mathbb{C}((\varepsilon))$

An element $x \in \widehat{\mathfrak{g}}$ is elliptic reg s.s. if it is
- topologically nilpotent
- regular semisimple
- $Z_{\widehat{G}}(x)$ is an anisotropic torus : no cocharacters

$$X_*(Z_{\widehat{G}}(x)) = 0$$

Fix $c = \frac{k}{m}$ $\quad h_c = (\rho^\vee(\tau^c), \tau, \tau^{c/2}) \in H$

There is an action of $\widehat{G} \times \mathbb{C}^*$ on $\widehat{\mathfrak{g}}$ by

$\tilde{g} = (g, \tau, \xi) : \tilde{g} \cdot x = \xi^{-2} (\mathrm{ad}\, g \circ F_\tau \cdot x)$

$$F_\tau \; a(\varepsilon) = a(\tau\varepsilon) \quad \text{loop rotation}$$

Fact: $\mathfrak{g}^{h_c}_{RS} = \emptyset$ iff m is a regular number

(K-L) $\mathfrak{g}^{h_c}_{Ell.RS} \neq \emptyset$ iff m is an elliptic number & $k>0$

- have fixed points for $h_c \iff$ properties of m.

Affine Flag Variety

$\mathcal{B} = \{$Iwahori subalgebras of $\tilde{\mathfrak{g}}\} = \tilde{G}\cdot\mathfrak{b}$

(can view inside Kashiwara flag variety)

$$\dot{\mathcal{N}} = \{(x, \mathfrak{b}') \in \mathfrak{g}\times\mathcal{B} : x\in\mathfrak{b}',\ x\in\mathcal{N}\}$$
$$= \tilde{G}\times_B \mathfrak{u}$$

$\pi : \dot{\mathcal{N}} \to \mathcal{N}$ (K-L)

$\exists$ a natural action of $\tilde{G}\times\mathbb{C}^*$ on $\dot{\mathcal{N}}$

$$\cup$$
$$\tilde{T} = T_0 \times \underset{q}{\mathbb{C}^*_r} \times \mathbb{C}^*_{\omega_0} \times \underset{t}{\mathbb{C}^*}$$

Fix $h \in \tilde{T}$

<u>Notation</u> $\tilde{G}^h = Z_{\tilde{G}}(h)$, $\tilde{\mathfrak{g}}^h$ same for Lie alg

<u>Lemma</u> i) $\tilde{G}^h$ is finite dimensional, connected, reductive
ii) $\tilde{\mathfrak{g}}^h$ is finite dimensional

Root system of $\tilde{G}^h, \tilde{T}$:
$$\Delta_h = \{\alpha \in \Delta : \alpha(s,\tau) = 1\} \text{ is finite}$$

For $\tilde{\mathfrak{g}}^h$: $\{\alpha \in \Delta \mid \alpha(s,\tau) = \xi^2\}$, finite.

<u>Facts</u> (K-L) - $\mathcal{N}^h$ is finite dimensional
- $\mathcal{B}^h$ is smooth, locally of finite type
$$\mathcal{B}^h = \bigcup_w \mathcal{B}^h_w \quad \text{smooth fin dim pieces}$$

where $\mathcal{B}^h_w$ = component containing standard Iwahori $\mathcal{B}_w$.

- $\dot{\mathcal{N}}^h$ is smooth, locally of finite type
$$\dot{\mathcal{N}}^h = \bigcup_w \dot{\mathcal{N}}^h_w$$

- $\dot\pi^h : \dot{\mathcal{N}}^h \longrightarrow \mathcal{N}^h$ proper map
- $(\pi^h)^{-1}(x) = \mathcal{B}^h_x$ is locally of finite type
- $\pi^{-1}(x) = \mathcal{B}_x$ is locally of finite type iff x is regular semisimple
- $\mathcal{B}_x$ is finite dimensional & connected iff x is elliptic regular semisimple (fin. many irreducible components)

<u>Proposition</u> For $x \in \mathcal{N}^h$, $H_*(\mathcal{B}^h_x)$ is finite dimensional iff x is elliptic regular semisimple

<u>Proposition</u> • If $k<0 \Rightarrow \forall\, x \in \mathcal{N}^h$ is <u>nilpotent</u> & $\exists$ fin many $\widetilde{G}^h$-orbits

• If $k>0 \Rightarrow \mathcal{N}^h = \mathfrak{g}^h$ but could have ∞ many $\widetilde{G}^h$-orbits

"Proof" : $x \in \mathcal{N} \Longleftrightarrow$ all invariant polys of x satisfy $\varphi_i(x) \in \varepsilon\, \mathbb{C}[[\varepsilon]]$

$x \in \mathcal{N}^h \Rightarrow \qquad \varphi_i(h.x) = \varphi_i(x)$

$\iff \varphi_i(F_\tau(x)) = F_\tau\, \varphi_i(x) = \tau^{>0} \varphi_i(x)$

(loop rotate by τ)

but OTOH if $x^h = x \Rightarrow \varphi_i(F_\tau(x)) = \varphi_i(\xi^2 x)$

$= \xi^{2d_i} \varphi_i(x)$

$\tau^{>0} = \xi^{>0}$ impossible if $k<0$

$\Rightarrow \varphi_i(x) = 0 \quad \forall i \quad \Delta$ x is nilpotent ... $\blacksquare$

<u>K-theoretic construction</u>

$\ddot{\mathcal{N}} = \{(x, b', b'') : x \in b' \cap b''\}$ "Steinberg"

$\rightarrow$ Convolution product on

$$\hat{K}(\ddot{\mathcal{N}}^h) = \prod_v \bigoplus_w K(\ddot{\mathcal{N}}^h_{v,w})$$

completion of K-groups according to components.

same for $H_*(\ddot{\mathcal{N}}^h)$ Borel-Moore homology

$\exists$ action of $\hat{K}(\ddot{\mathcal{N}}^h)$ on $K(\ddot{\mathcal{N}}^h)$ & $K(\mathcal{B}_x^h)$

Define $\hat{H}$ as $\underline{H}$ but with X_0 replaced by $X_0 \oplus \mathbb{Z}\omega_0$ - make tori are bigger!

$$\hat{H} = H \rtimes \mathbb{C}[X_{\omega_0}^{\pm 1}]$$

<u>Theorem</u> (Vasserot) $\exists \phi : \hat{H} \longrightarrow \hat{K}(\ddot{\mathcal{N}}^h)$

$\rightsquigarrow$ geometric classification of irreducible $\hat{H}$-modules

$\mathbb{C}_{\mathcal{N}^h}$ constant sheaf (shifted by dim on each component!)

$$L_h = \pi_*(\mathbb{C}_{\mathcal{N}^h}) = \bigoplus_{\chi} L_{h,\chi} \otimes IC(\chi)$$

χ irred local systems on locally closed subvarieties of $\mathcal{N}_h$

$\chi_h = \{$irred local systems which appear$\}$

(h, χ) is a <u>Langlands parameter</u>

Theorem (Vasserot) i) $L_{h,\chi} \in \operatorname{Irr}(\mathcal{O}_h(\hat{H}))$

ii) This exhausts all the irreducibles

iii) $L_{h,\chi} \cong L_{h',\chi'}$ iff $(h,\chi), (h',\chi')$ $\tilde{G}$-conjugate

iv) In Grothendieck group

$$H_*(\mathcal{B}_x^h) = \sum_{\chi} L_{h,\chi} \otimes IC(\chi)|_x$$

as $\hat{H}$-modules.

Problem Parametrize $\mathcal{X}_h$

Assume $k<0$ => fin many orbits in $\mathcal{N}^h$

$x \in \mathcal{N}^h$ => $\pi_1(\mathrm{Orb}_x) = \pi_0(Z_{\tilde{G}^h}(x))$ acts on Springer fiber

$= A(h,x)$

Theorem $\mathcal{X}_h = \{\sigma \in \operatorname{Irr}(A(h,x))$ arising in $H_*(\mathcal{B}_x^h)$ as x varies in $\mathcal{N}^h\}$

Corollary ($k<0$) $\mathbb{C}_{\{0\}} \in \mathcal{X}_h$

How about $k > 0$?

Use Fourier transform between $\mathcal{N}^h$ for $k > 0$ & $\mathcal{N}^h$ for $k < 0$: Involution on set of local systems that appear

Fix a nondegenerate pairing $\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$

$$h \longmapsto h^\# \qquad (s, \tau, \zeta) \mapsto (s, \bar{\tau}, \zeta^{-1})$$

$(,)$ restricts to a nondegen pairing

$$\mathfrak{g}^h \otimes \mathfrak{g}^{h^\#} \longrightarrow \mathbb{C}$$

$\Rightarrow$ have Fourier-Sato transform

$$FS : D^b_{\mathbb{R}^+}(\mathfrak{g}^h) \simeq D^b_{\mathbb{R}^+}(\mathfrak{g}^{h^\#})$$

on conic sheaves

$$\Rightarrow D^b_{\tilde{G}^h \times \mathbb{C}^*}(\mathfrak{g}^h) \simeq D^b_{\tilde{G}^h \times \mathbb{C}^*}(\mathfrak{g}^{h^\#})$$

$$k > 0 \longmapsto k < 0$$

$$FS(\mathbb{C}_{\mathfrak{g}^h}[\dim \mathfrak{g}^h]) = \mathbb{C}_{\{0\}}$$

Theorem (Vasserot) $\exists$ a bijection

$\chi_h \longleftrightarrow \chi_{h^*}$ s.t.

$FS(IC(\chi)) = IC(\chi^*)$

.... analog of result of Lusztig & Mirkovic on character sheaves

Use to take $h>0$ to $h<0$.

Thm (V-V) The polynomial module P^h_x is realized as $H_*(\mathcal{B}^h_\circ)$ ...

Show it has simple top, & that top is fin dim iff x is elliptic — by Fourier transform!

$\Rightarrow L_{h_c, \mathbb{C}_{\{0\}}}$ is spherical & f.d if $k<0$ & m elliptic