Olivier Schiffmann - Crystal structures on some global nilpotent cones

Note Title 4/10/2008

Idea: project to export some techniques from quivers to curves

(related: Crawley-Boevey on Deligne-Simpson; Hausel on Betti # of Higgs moduli)

X smooth proj curve $/\mathbb{F}_q$

H_X : Hall algebra of $Coh(X)$

$$H_X = \bigoplus_{r,d} \{ H_X[r,d] = \{ f : \underline{Coh}^{(r,d)} \to \mathbb{C} \} \}$$

Multiplication: $$fg(\mathcal{H}) = \sum_{K \subset H} v^{\langle H/K, K \rangle} f(H/K)\, g(K)$$

where $\langle H, K \rangle = \dim \mathrm{Hom} - \dim \mathrm{Ext}^1$

$v = q^{-\frac{1}{2}}$

Comultiplication: $$\Delta(f)\,(H,K) = \frac{v^{-\langle H,K \rangle}}{\# \mathrm{Ext}^1} \sum_{\xi \in \mathrm{Ext}^1(K,H)} f(M_\xi)$$

Bilinear form: $\langle 1_F, 1_G \rangle = \delta_{F,G} / \# \mathrm{Aut}\, F$

Theorem (Green)

i) (H_X, m, Δ) is a (twisted) bialgebra

ii) $\langle , \rangle$ is a Hopf pairing

Remark $\mathrm{Coh}\, X \rightsquigarrow \mathrm{Rep}\, Q$

$\Rightarrow H_X \supset U_q^+ \mathfrak{g}_Q$ positive half of quantum group

Remark Primitive elements of $H_X \longleftrightarrow$ simple skyscraper sheaves $\Rightarrow$ recover X from H_X.

Natural subalgebra ("spherical subalgebra")

$U_X^+ \subset H_X$: generated by characteristic functions $1_{r,d} = 1_{\mathrm{Coh}^{r,d}(X)} \in H_X$

(suffices to take $r \leq 1$)

Properties i) Semistable char. fns $1^{ss}_{r,d} \in U_X^+ \quad \forall r,d$

ii) U_X^+ is stable under Δ.

$U_X :=$ Drinfeld double of U_X^+

<u>Examples</u> • $X = \mathbb{P}^1 \Rightarrow U_X = U_v(\widehat{sl}_2)$
(in Drinfeld presentation)
-- U_X^+ is a deformation of
$U_v(\mathfrak{h}[t] \oplus e \otimes \mathbb{C}[t,t^{-1}])$

• $X = E$ elliptic curve : $U_X = U_{v,t}(\mathcal{E}\mathfrak{gl}(1))$
$= \ddot{SH}_{GL(\infty)}$

where $\mathcal{E}\mathfrak{gl}(1) = \mathbb{C}[t^{\pm 1}, s^{\pm 1}]$ elliptic Gl_1

$\longleftrightarrow$ (Schur-Weyl) ∞-rank Cherednik DAHA

<u>Remark</u> U_X or U_X^+ depends only on genus g,
$q = v^{-2}$ & $\{\alpha_1, \dots \alpha_{2g}\}$ Frobenius eigenvalues on $H^1(X, \bar{\mathbb{Q}}_\ell)$
(follows from presentation as quotient of a free algebra by a bilinear form)
~ hence $\exists$ a universal algebra $U_{g,v,\alpha_1,\dots\alpha_g}$.

Geometric version $\quad \bar{X} = X \otimes \bar{\mathbb{F}}_q$

$\underline{Coh}\ \bar{X} = \coprod_{r,d} \underline{Coh}^{r,d}(\bar{X})$ smooth stack, locally of finite type

$$\begin{tikzcd}
 & \mathcal{E} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\underline{Coh}^{r,d} \times \underline{Coh}^{r',d'} & & \underline{Coh}^{r+r',d+d'}
\end{tikzcd}$$

Ext correspondence $\Rightarrow$ defines convolutions:

Ind : $p_{2*} p_1^* : D^b(\underline{Coh}^{r,d} \times \underline{Coh}^{r',d'}) \longrightarrow D^b(\underline{Coh}^{r+r',d+d'})$

Res : $p_{1*} p_2^*$.

Def: $\mathbb{1}_{r,d} = \bar{\mathbb{Q}}_\ell |_{\underline{Coh}^{r,d}} [\dim \underline{Coh}^{r,d}]$

$\mathcal{P}$ = simple perverse sheaves arising in
$\mathbb{1}_{r_1,d_1} \star \cdots \star \mathbb{1}_{r_k,d_k}$ with $r_i \leq 1$, $d_i \in \mathbb{Z}$.

Conjecture i) $\{ \mathrm{Tr}\, \mathbb{P} : \mathbb{P} \in \mathcal{P} \}$ is a basis of U_X^+.
(for Frobenius-fixed $\mathbb{P}$ .... maybe all are)

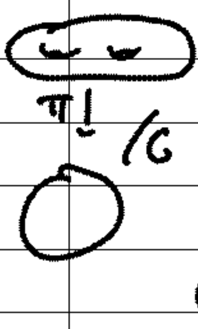
ii) All $\mathbb{P}$ are very pure : Frob eigenvalues lie in $\alpha_1^{\mathbb{Z}} \alpha_2^{\mathbb{Z}} \cdots \alpha_g^{\mathbb{Z}} v^{\mathbb{Z}}$)

& pointwise pure

iii) $\exists$ 1-1 correspondence with irred components of global nilpotent cone Λ_X for moduli of coherent sheaves

Remark: Conjecture OK for $g \leq 1$.

<u>Possible approach</u> : Base change to ramified genus & find X s.t. $\exists\, G \subset \mathrm{Aut}\, X$, $X/G \simeq \mathbb{P}^1$



π ramified along $\{\lambda_1 \ldots \lambda_N\} \in \mathbb{P}^1$ with indices $\{p_1 \ldots p_N\} \in \mathbb{N}_{>1}$.

$\mathrm{Coh}_G X$ can be described by linear algebra data

– as category of D-parabolic coherent sheaves on $\mathbb{P}^1$ ($D = \sum p_i \lambda_i$) $\quad \mathrm{Coh}_G X = \mathrm{Coh}_D \mathbb{P}^1$

A D-parabolic vector bundle is $(V, F_{i,j})$

with $\forall i$: $F_{i,1} \subset F_{i,2} \subset \dots \subset F_{i,j} = V|_{x_i}$ flags

A D-parabolic coherent sheaf is $(F, F_{i,p}, \varphi_{i,p})$

with $F \xrightarrow{\varphi_{i,1}} F_{i,1} \xrightarrow{\varphi_{i,2}} F_{i,2} \to \dots \to F_{i,p_i} = F(x_i)$

s.t. $\varphi_{i,p_i} \circ \dots \circ \varphi_{i,1} : F \longrightarrow F(x_i)$ is

a natural map.

Point. i) $Pic(X,G)$ is discrete

$\simeq \mathbb{Z}x_1 \oplus \dots \oplus \mathbb{Z}x_N \,/\, p_i x_i = p_j x_j$

$Pic(X,G) / \langle \mathcal{O}(G\cdot x) \text{ free orbit} \rangle \simeq \prod \mathbb{Z}/p_i\mathbb{Z}$

i.) Line bundles are rigid: $Ext^1(L,L) = \{0\}$

ii) Torsion sheaves: $x \in \mathbb{P}^1 \setminus \{x_i\} \Rightarrow$ free G-orbit

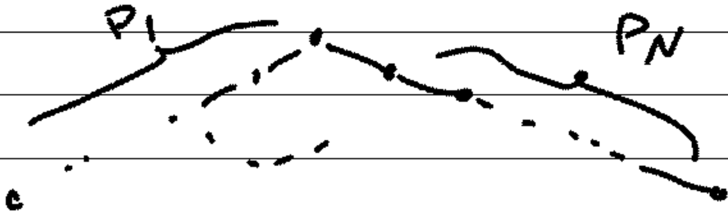
$\Rightarrow$ $\exists$ a unique simple torsion sheaf

- $x = x_i$: $G\cdot x$ has stabilizer $\mathbb{Z}/p_i$

  p_i simple torsion sheaves

4. $K_0(\mathrm{Coh}_G(X)) \simeq \dfrac{\mathbb{Z}[\mathcal{O}] \oplus_{i,p} \mathbb{Z}[\mathcal{O}^{(p)}_{x_i}]}{\sum_{1}^{p_i}[\mathcal{O}^{(0)}_{x_i}] = \sum^{p_j}[\mathcal{O}^{(p)}_{x_j}]}$

Associate to (X,G) a star-shaped quiver



Define $H_{X,G}$, $U^+_{X,G}$, ... in same way

<u>Theorem</u> (Schiffmann)

$$U^+_{X,G} = U_v^+(L\mathfrak{g}_p)$$

$\mathfrak{g}_p$ = K-M algebra, $L\mathfrak{g}_p$ - loop algebra

$$0 \to K \to L\mathfrak{g}_p \to \mathfrak{g}_p[t,t^{-1}] \to 0$$

(Moody-Rao).

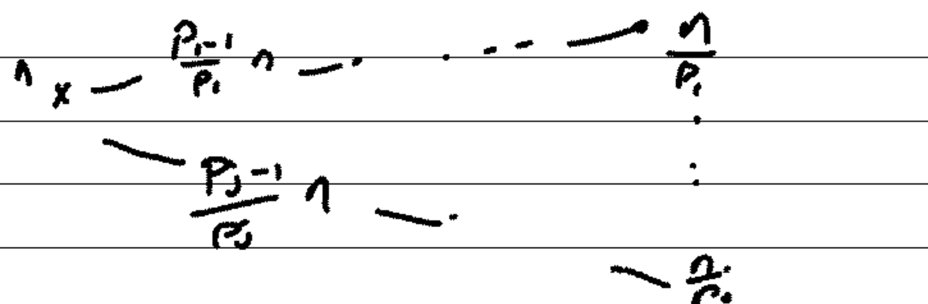
Naive idea: relate U_X^+ to $U^+_{X,G}$ using equivariantization

On K_0 this gives

$$\mathbb{Z}^2 = K_0'(\mathrm{Coh}\, X) \longrightarrow K_0(\mathrm{Coh}_G\, X)$$

which maps into imaginary weight spaces.

ex. $[\mathcal{O}_X]$ goes to

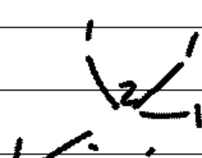


which is imaginary if Γ is not of finite type

$[\mathcal{O}_x] \rightsquigarrow \delta$ imaginary root

- $G = \mathbb{Z}/2 \curvearrowright X$ hyperelliptic

$\Rightarrow$ $2g+2$ branches



& δ (imaginary) span a $\mathbb{Z}^2$-lattice in the root system $\Delta(L_{g,p})$

- X elliptic, $G = \mathbb{Z}/i\mathbb{Z}$ $\quad i = \{2,3,4,6\}$

  $\Gamma = D_4^{(1)}, E_n^{(1)}$ affine algebras $(n=6,7,8)$

  $L\mathfrak{g}_\Gamma = \mathcal{E}\mathfrak{g}$ for $\mathfrak{g} = D_4, E_n$ elliptic algebras

  $U_X \simeq U_v(\mathcal{E}\mathfrak{gl}_1) \quad : \mathcal{E}\mathfrak{gl}_1 \hookrightarrow \mathcal{E}\mathfrak{g}$.

<u>Crystals</u> Recall: Q a quiver, d dimension vector

E_d : space of reps of dim d $\circlearrowleft G_d$

$\mathcal{M}_d = E_d/G_d$.

$T^*\mathcal{M}_d = \mu^{-1}(0)/G_d$, $\quad \mu : T^*E_d \longrightarrow \mathfrak{g}_d^*$
moment map

$\Lambda_d \subset T^*\mathcal{M}_d$: Lagrangian substack of nilpotent reps

1. $\coprod_d \operatorname{Irr} \Lambda_d$ (irred components) has structure of an I-colored graph
   $\cdot\cdot$ I = vert (Q) = simple roots

- can refine to a quiver [Lusztig]

2. [Kashiwara - Saito]

$\coprod_{\alpha} \mathrm{Irr}\, \Lambda_\alpha \overset{1-1}{\longleftrightarrow} \mathcal{P} = \coprod \mathcal{P}_\alpha$

simple perverse sheaves on E_α giving Lusztig canonical basis.

$\Rightarrow$ I-colored graph structure on the canonical basis: crystal graph of $U^+\mathfrak{g}$

For us: curve $X \circlearrowleft G$

$\alpha \in K_0(\mathrm{Coh}_G X) \rightsquigarrow \underline{\mathrm{Coh}}^\alpha$ moduli stack of coherent sheaves of class α

$T^*\underline{\mathrm{Coh}}^\alpha = \underline{\mathrm{Higgs}}^\alpha \supset \Lambda_\alpha$: global nilpotent cone

Prop Assume $X = \mathbb{P}^1 \circlearrowleft G$ ($\Longleftrightarrow$ quiver Γ of finite type) Then $\Lambda = \coprod_{v,\tau,\lambda} T^*\overline{U_{v,\tau,\lambda}}$

(closed) union of conormal bundles to strata, where
V = vector bundle, τ = torsion sheaf supported on D & λ = partition

$$U_{V,\tau,\lambda} = \{ V \oplus \tau \oplus \mathcal{O}_{x_1}^{(\lambda_1)} \oplus \dots \oplus \mathcal{O}_{x_\ell}^{(\lambda_\ell)} \}$$
$$x_1, \dots, x_\ell \in \mathbb{P}^1 \setminus \{\lambda_i\}$$

Here have no simple objects but will use rigid objects — namely line bundles — to construct crystal structure: get arrows labelled by $\mathrm{Pic}(X, G)$.

Note! L rigid, $H^0(\Omega_{X,D}) = 0 \Rightarrow$ no deformations of trivial Higgs field on L.

$$\Lambda^\alpha_{L,\geq r} = \{ (\mathcal{F}, \varphi) : \exists \, (L^{\oplus r}, 0) \hookrightarrow (\mathcal{F}, \varphi) \}$$

$$\Lambda^\alpha_{L,r} = \Lambda^\alpha_{L,\geq r} \setminus \Lambda^\alpha_{L,\geq r+1}$$

Correspondences

$$\Lambda^\alpha_{L,r} \xleftarrow{\pi_1} \mathcal{E}^\alpha_{L,r} \xrightarrow{\pi_2} \Lambda^{\alpha - r\bar{L}}_{L,0}$$

$$\mathcal{E}^{\alpha}_{L,r} = \{ (\mathcal{G}, F, \varphi) \mid (\mathcal{G}, \varphi|_{\mathcal{G}}) \cong (L^{\oplus r}, 0)$$
$$\nexists\, (L,0) \hookrightarrow (F/\mathcal{G}, \varphi) \}$$

<u>Proposition</u> i) π_2 is <u>stratified</u> affine fibration
ii) fibers of π_1 are smooth, irred & locally constant along a stratification.

So $$\mathrm{Irr}\,\Lambda^{\alpha}_{L,r} \underset{i_r^*}{\overset{j_r^{\alpha-1}\ \sim}{\rightleftarrows}} \mathrm{Irr}\,\Lambda^{\alpha-L_r}_{L,0}$$

$\Rightarrow$ construct crystal structure on $\mathrm{Irr}\,\Lambda$

$$\begin{cases} \tilde{e}_L = j_\alpha^{r-1} \circ i_\alpha^{r} \\ \tilde{f}_L = j_\alpha^{r+1} \circ i_\alpha^{r} \end{cases}$$

<u>Example</u> $\mathbb{P}^1$, $\widehat{\mathfrak{sl}}_2$ ($\frac{\infty}{2}$ crystal graph of $U_v^+\widehat{\mathfrak{sl}}_2$)

Pieces of crystal graph:

$$\begin{tikzcd}
 & \phi \arrow[dl] \arrow[dr, "f_{\mathcal{O}}"] & & (1) \arrow[dl, "f_{\mathcal{O}(-1)}"'] \arrow[dr] & \\
\mathcal{O}(-1) \arrow[dr] & & \mathcal{O} \arrow[dl] \arrow[dr] & & \mathcal{O}(1) \arrow[dl] \\
 & \mathcal{O}\oplus\mathcal{O}(-1) & & \mathcal{O}\oplus\mathcal{O} &
\end{tikzcd}$$

(1^2) $\qquad\qquad\qquad\qquad$ (2)

$\swarrow \mathcal{O}(-1)$ $\qquad\qquad\qquad\qquad$ $\downarrow \mathcal{O}$

$\mathcal{O}(1) \quad \mathcal{O}(3)$ $\qquad\qquad\qquad$ $\mathcal{O}(1), (1)$

$\qquad\qquad\qquad\qquad\qquad\qquad$ $\swarrow \mathcal{O}(2)$

$\longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(3)$

ϕ

$\swarrow \mathcal{O}(0)$

$\mathcal{O}(0)$

$\swarrow \mathcal{O}(2)$

$\mathcal{O}(0) \oplus \mathcal{O}(2)$

$\mathcal{O}(0) \oplus \cdots \mathcal{O}(2n)$

subgraph corresponding to all stable Higgs bundles as quotient of

$$\left(\bigoplus_{0}^{n} \mathcal{O}(2i), \varphi\right)$$

φ nondegenerate:

analog of Nakajima Lagrangian Quiver variety! gives crystal graph L_n of $U_v(\widehat{sl}_2)$

Hausel's conjecture

$\underline{Higgs^{\alpha, st}_{(X,G)}}$ = stable Higgs bundles

$\alpha = \alpha' + l\delta$ for α' a dim. vector for Γ

$a_{\alpha'}(q)$ = Kac's polynomial for Γ :
= # absolutely indecomp. reps of class α' over $\mathbb{F}_q$

Kac conjecture : $a_\alpha(0) = \dim \mathfrak{g}_\Gamma[\alpha']$
Hausel proved this by showing
$a_\alpha(0) = \# \mathrm{Irr}(\Lambda_0^{\alpha, st})$ fix underlying vector bundle to be trivial

$\Lambda_0^{\alpha, st} = (\mathcal{F}, \varphi)$: underlying bundle on $\mathbb{P}^1$ is trivial
(Nakajima quiver variety)

Conjecture (Hausel) $\#\mathrm{Irr}(\Lambda^{\alpha, st}) = a_\alpha(1)$

... expect filtration on $\mathrm{Irr}(\Lambda^{\alpha, st})$ capturing full polynomial a_d — coming from crystal...