C. Stroppel - Representation Theory of Link Homology

Note Title 2/5/2008

1. Khovanov's algebra, revisited / generalized
2. Categorification / link invariants
3. Degenerate affine Hecke algebra
4. Lie supergroup $GL(m|n)$

Fix $n = 2k \in \mathbb{Z}_+$ even

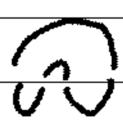
$\mathrm{Cup}(n) = \{$crossingless matchings of n points$\}$

eg $n=4$  a ,  b

$\Rightarrow$ Khovanov's arc algebra $H_n = \bigoplus_{a,b \in \mathrm{Cup}(n)} aH_nb$ as vector space

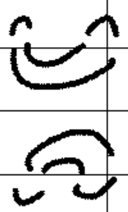
($a,b \rightsquigarrow$ idempotents in algebra)

$aH_nb =$ Hom between corresponding indecomposable projectives

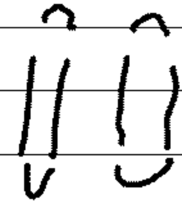
$\frac{\bar{b}}{a}$  circle diagram $\rightsquigarrow \left(\mathbb{C}[x]/x^2\right)^{\otimes \#\text{circles in } \frac{\bar{b}}{a}}$

(ie apply 2d TFT associated to the Frobenius algebra $H = \mathbb{C}[x]/x^2$

<u>Multiplication</u> $aH_nb \otimes bH_nc \longrightarrow aH_nc$:



$$\begin{matrix}\bar{c}\\ b\\ \bar{b}\\ a\end{matrix} \Rightarrow \text{minimal cobordism} \Rightarrow \begin{matrix}\bar{c}\\ a\end{matrix}$$

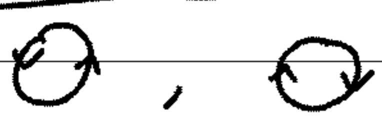


gives linear map via 2d TFT

$\xrightarrow{m}$  $\xrightarrow{\Delta}$

H_n has a $\mathbb{Z}$-grading & this multiplication is commutative.

<u>Reformulation</u> Consider oriented circles and



use them to label the two basis vectors in $\mathbb{C}[x]/x^2$

$1 \in \mathbb{C}[x]/x^2$ $\quad$ $x \in \mathbb{C}[x]/x^2$

$\Rightarrow$ as vector space $H^n = \bigoplus_{a,b}$ oriented diagrams of form $\begin{matrix}\bar{b}\\ a\end{matrix}$

$= \bigoplus_{a,b,\lambda} \left\langle \begin{matrix}\bar{b}\\ \lambda\\ a\end{matrix} \right\rangle$ where λ runs through all orientations (considered as sequences of ups $\wedge$ & downs $\vee$)

$\deg \overset{\bar b}{\underset{a}{\lambda}} = \#$ clockwise cups/caps

e.g.  $\rightsquigarrow 0$

 $\rightsquigarrow 2$

Lemma (Cellularity) $(a\lambda b)(c\mu d) = \begin{cases} 0 & \text{if } b \neq \bar c \\ \gamma(a\mu d) + \text{bigger} & \text{if } (a\mu d) \text{ is oriented} \\ \text{bigger} & \text{otherwise} \end{cases}$

where "bigger" means it's a linear combination of basis vectors and where $\nu > \mu$
& partial ordering is generated by $\wedge\vee > \vee\wedge$ [call $\wedge\vee$ sequences weights]

$\gamma \in \{0,1\}$, independent of d.

Cell module $M(\lambda) = \text{Span}\langle \underbrace{a}_{\text{oriented cup diagram}}\lambda | \rangle$

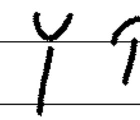
Corollary: Khovanov's algebra is cellular.
(but not quasi-hereditary: it's Frobenius/symmetric, so self-injective, so not finite global dimension.)

Q: What (if it exists) is a quasi-hereditary cover? (Rouquier)

Generalization

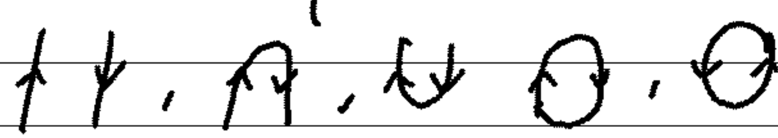
Let a *weight* be a labelling of vertices in the number line by $\circ, \times, \wedge, \vee$, with no $\vee\wedge$ outside a finite set

e.g. $\wedge \ \wedge \ \times \ \circ \ \wedge \ \vee \ \vee \ \times \ \circ \ \vee \ \vee$

Oriented cup diagram: points are only $\vee$'s & $\wedge$'s.
& cups allow also rays, but no 

block := set of weights related by swaps of $\wedge\vee \longleftrightarrow \vee\wedge$

Example: Take block $\{\wedge\vee, \vee\wedge\}$

$\Rightarrow$ basis 

algebra of principal block of Category $\mathcal{O}$ for sl_2, with Koszul grading

Fix block $\Lambda_{m,n}$ with m $\wedge$'s, n $\vee$'s

$\Rightarrow$ algebra $K^{m,n}$

$K_0(K^{m,n}\text{-mod}) = K_0(\mathrm{Perv}_B(Gr(m,m+n)))$

$= K_0(\mathcal{O}^{m,n}(\mathfrak{gl}_{m+n}))$ parabolic block

cell module $[M(\lambda)] \longleftrightarrow [i_! \mathbb{C}_{X_\lambda}]$

$\longleftrightarrow$ [parabolic Verma modules $\Delta(\lambda)$]

(λ gives a Young diagram)

Simples $[S(\lambda)] \longleftrightarrow$ [simple IC $i_{!*} \mathbb{C}_{X_\lambda}$]

$\longleftrightarrow$ simple highest wt modules

$\Delta(\lambda)$ has highest weight λ

$\rho = (m+n, m+n-1, \dots, 1)$

<u>Theorem</u> (Stroppel; based on conjectures of Khovanov, Braden)

$K^{m,n}\text{-mod} \simeq \mathcal{O}_0^{m,n}(\mathfrak{gl}(m+n))$

Categorification

1. $U_q\mathfrak{sl}_2$ tangle invariants:

V = natural representation

standard basis of $V^{\otimes k}$ (ups & downs: spin basis) $\longleftrightarrow \left[\bigoplus_{j=0}^{k} K^{j,k-j}\text{-mod}\right]$

So get categorifications of $V^{\otimes k}$ from $K^{m,n}$-mod, $\mathcal{O}^{m,n}(\mathfrak{gl}_{m+n})$ or $\mathrm{Perv}_B(Gr)$

Now use projective functors (tensor by finite dim reps) on category $\mathcal{O}$

<u>Theorem</u> (Stroppel; partly conjectured by Bernstein-Frenkel-Khovanov)

a. (Split) Grothendieck group of projective functors on $\bigoplus \mathcal{O}_0^{k,j-k}(\mathfrak{gl}(k))$ is isomorphic to the Temperley-Lieb algebra ($\longleftrightarrow$ tangles without crossings)

generators $|\,|\cdots|\ \underset{i}{\overset{\smile}{\frown}}\ |\cdots\underset{k}{|}$ $\longleftrightarrow$ translation through ith wall

[... really passes through singular categories, with dummy diagrams $\;||\,\substack{\cup \\ \circ\times \\ \cap}\,||$]

b. Can be extended to tangle invariants
c. Restricts to Khovanov's theory.

To get crossings: have natural transformations (by adjunction) from Id to translation functor $\;|| \longrightarrow \substack{\cup \\ \cap}\;$ $\Rightarrow$ take cone;
Braid group acts by intertwining functors.

2. $U_q\mathfrak{sl}_k$: V = natural representation,
want to categorify $\Lambda^{r_1}V\otimes\ldots\otimes\Lambda^{r_s}V$
& intertwiners. $\cong K\left[\bigoplus_{\mu}\mathcal{O}^{\mu}_{\underline{r}}(\mathfrak{gl}(t))\right]$

$t=\sum r_i$ multi-index $\rightsquigarrow$ μ with at most k parts

$\underline{r} \Rightarrow$ singularity of the blocks we take:

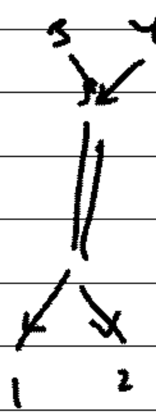
$\mathcal{O}_r \longrightarrow$ stabilizer is $S_{r_1}\times S_{r_2}\times\dots\times S_{r_s}$.

Sussan: projective functors give intertwiners
--- in Koszul dual picture.
— but above picture is much easier, really reduces to Hecke algebra calculations

Conjecture: this theory "restricts" to Khovanov-Rozansky.

<u>Example</u>

$$V\otimes V \to \Lambda^2 V \to V\otimes V$$



K-R $\rightsquigarrow$ matrix factorization

$$\begin{pmatrix} x_1+x_2-x_3-x_4 \\ x_1x_2-x_3x_4 \end{pmatrix}$$

... tensor product of $R \xrightarrow{\Sigma \pm x} R$, $R=\mathbb{C}[x_1,\dots,x_4]$

$R \xrightarrow[x_1x_2-x_3x_4]{} R$

Cohomology $\mathbb{C}[x_1,x_2]\otimes_{\mathbb{C}[x_1,x_2]^S}\mathbb{C}[x_1,x_2]$ S = permutation.

Rep. theory origin:

$$
\begin{tikzcd}
\text{res block} \arrow[d, "\text{onto wall}"] & \mathcal{O}_0^{(1,1)}(\mathfrak{gl}_2) \arrow[r, "\mathbb{V}"] \arrow[d] \arrow[rr, bend left, leftrightarrow, "\text{Koszul duality}"] & H^\bullet(G/B)\text{-mod} \arrow[d, "\text{res}"] & \mathrm{Perv}_B(G/B) \arrow[l, "\mathbb{H}^\bullet"'] \arrow[d] \\
\text{sing block} \arrow[d, "\text{off wall}"] & \mathcal{O}_{-\rho}^{(1,1)}(\mathfrak{gl}_2) \arrow[r, "\mathbb{V}"] \arrow[d] & H^\bullet(G/P)\text{-mod} \arrow[d, "\text{ind}"] & \mathrm{Perv}_B(G/P = \bullet) \arrow[l, "\mathbb{H}^\bullet"'] \arrow[d] \\
\text{res block} & \mathcal{O}_0^{(1,1)}(\mathfrak{gl}_2) \arrow[r, "\mathbb{V}"] & H^\bullet(G/B)\text{-mod} & \mathrm{Perv}_B(G/B) \arrow[l, "\mathbb{H}^\bullet"']
\end{tikzcd}
$$

$\mathbb{C}[x_1,x_2] \otimes_{\mathbb{C}[x_1,x_2]^S} \mathbb{C}[x_1,x_2]$ = exactly restrict & then induct, in deformed / equivariant picture

$\mathbb{V}$ = Soergel's functor

In this picture $\mathrm{Perv}_B(G/B)$ is the wrong geometry to underlie $\mathcal{O}(\mathfrak{gl}_2)$ ... it's Koszul dual... but rather should use coherent sheaves on Springer fibers as the geometry...

Fix $m, n \in \mathbb{Z}_+$ $m \geq n$

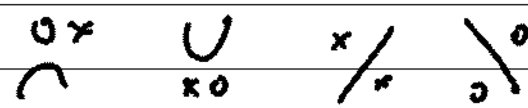
$\mathfrak{gl}(\infty) = \mathbb{Z} \times \mathbb{Z}$-matrices with only finitely many nonzero entries

V = natural module.

Can we categorify $\Lambda^m V \otimes \Lambda^n V$?

$\cong K_0\left(\bigoplus_{\Lambda} K^{\Lambda}\text{-mod} \right)$:

allow only blocks with r x's $(m-r)$ v's $(n-r)$ ∧'s all others are o. $0 \leq r \leq m$

Chevalley generators : 

E_i, F_i etc

$E := \bigoplus E_i$ $F = \bigoplus F_i$

Theorem (Brundan, Chuang-Rouquier)

a. There is an action of the degenerate affine Hecke algebra H_d on F^d by natural transformations

$[H_d = \mathbb{C}[x_1 \ldots x_d] \otimes \mathbb{C}[S_d]$ smash product ...$]$

particular projective which is
module, single

b. $H_d \twoheadrightarrow \mathrm{End}\left(F^d P(v \cdot v \times\times \;\; \overset{d}{o o \cdots o})\right)$

with image the cyclotomic Hecke algebra
of level 2: factor out $f(x_1) = (x_1 - m)(x_1 - n)$

$\rightsquigarrow$ in principle this cyclotomic Hecke algebra
should control category $\mathcal{O}$...

4. Super Lie group $G := GL(m|n)$

$G \supset B \supset T \quad X(T)$ weights $\supset X(T)^+$

$\lambda \in X(T)^+ \rightsquigarrow M(\lambda)$ Kac module

$\downarrow$
$L(\lambda)$ simple module

$\rightsquigarrow$ construct infinite weight on the number line

$\mathcal{F}_{m|n}$ = category of fin dim $GL(m|n)$ modules

Conjecture (Brundan-Stroppel)

$\mathcal{F}_{m|n} \simeq \bigoplus_{\Lambda} K^{\Lambda}\text{-mod}$

$\dashrightarrow$ blocks of weights given by $X(T)^+$

... combinatorics works!

Would imply K is Koszul : Khovanov's algebra
& its quasihereditary cover agree ...
inf dim symmetric quasi hereditary

$\rightsquigarrow$ get geometric picture for $GL(m|n)$ modules
from perverse sheaves on Grassmannians
in the limit.

($K^{\infty,n}$ algebra : $\mathcal{O}^{m,n}(\mathfrak{gl}(\infty+n))$

(Brundan: multiplicities of simples in Kac modules
given by Kazhdan-Lusztig polynomials)
... in this simple setting all multiplicities
are 0 or 1..