B. Toën - Hall Algebras for triangulated categories

Note Title 3/4/2008

(and higher algebraic stacks)

$\mathcal{A}$ abelian category

1) categorical construction: $\mathcal{A}/\mathbb{F}_q$ & Hom, Ext^1 finite $\Rightarrow H(\mathcal{A})$

Hall algebra of $\mathcal{A}$

2) geometric construction:

eg $\mathcal{A} = B\text{-mod}^{fd}$ $\left\{ \begin{array}{l} \exists\, \mathcal{M}_{\mathcal{A}} = \text{moduli of objects in } \mathcal{A} \\ \text{as algebraic stack} \\ \mathcal{M}_{\mathcal{A}}^{\hookrightarrow} : \text{moduli for monomorphisms in } \mathcal{A} \end{array}\right.$

$\Rightarrow H^{geom}(\mathcal{A})$

1) $H(\mathcal{A})$ = free abelian group over $\mathcal{A}/\text{iso}$ (set of isom classes of objects), with standard basis $\{\chi_\alpha\}_{\alpha \in \text{Isom classes}}$. Associative product:

$$\chi_x \cdot \chi_y = \sum g_{xy}^z \, \chi_z$$

$$g_{xy}^z = \# \{ x' \subset Z \;:\; x' \simeq x,\ Z/x' \simeq y \}$$

2)
$$\begin{tikzcd} (x \hookrightarrow z) \arrow[r, mapsto] & z \\ \mathcal{M}_{\mathcal{A}}^{\hookrightarrow} \arrow[r, "c"] \arrow[d, "\pi"] & \mathcal{M}_{\mathcal{A}} \\ \mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} & \end{tikzcd}$$

$(x, z/x)$

$$c_! \pi^* : D(\mathcal{M}_{\mathcal{A}}) \times D(\mathcal{M}_{\mathcal{A}}) \longrightarrow D(\mathcal{M}_{\mathcal{A}})$$

associative tensor product on triangulated categories

$$H^{geom}(\mathcal{A}) = K_0(D(\mathcal{M}_{\mathcal{A}}))$$ with some finiteness conditions

2') Can use algebraic stacks over the moduli space, & make a Grothendieck group out of them

$$K_0(\text{alg stacks}/\mathcal{M}_{\mathcal{A}}) = H^{abs}(\mathcal{A})$$

absolute Hall algebra

Relations on K_0 (alg stacks $/\mathcal{M}_x$):

- $F_0 \hookrightarrow F_1 \hookleftarrow F_1 \setminus F_0$

  want $F_1 = F_0 + (F_1 - F_0)$ in K-groups

- For Zariski locally trivial fibration want to write class of total space as product of base by fiber

  (automatic for schemes, not for stacks..!)

Interrelation:

$$H^{abs}(A) \longrightarrow H^{geom}(\mathcal{B}) \longrightarrow H(A)$$

$$(F \xrightarrow{p} \mathcal{M}_x) \longmapsto p_! \, \mathbb{Q}_\ell$$

$$E \longmapsto \begin{pmatrix} \text{trace function} \\ \text{of Frobenius on } E \end{pmatrix}$$

Goal: Generalize this to triangulated categories.

Remark: naive generalization of Hall numbers

g_{xy}^{z} to triangulated cats $/\mathbb{F}_q$ doesn't work.

Suppose $T/\mathbb{F}_q$ with finite Hom, Ext^1.

Naive Hall:
$$g_{xy}^{z} = \frac{\# \text{ triangles } x \to z \to y \to x[1]}{\# \operatorname{aut}(x) \ \# \operatorname{aut}(y)}$$

$\longrightarrow$ defines non associative multiplication
on $\mathbb{Q}[T/\text{iso}]$

Problem: count triangles in T in more subtle way.

Suppose $T \simeq D_{fd}(B)$

B dg algebra $/\mathbb{F}_q$ "of finite type"
(fin many gens & relations up to retraction)

$D_{fd}(B)$ $\quad E : H^*(E)$ fin dim

- we'll give invariant of B up to derived Morita.

$H(T) := \mathbb{Q}[T/_{\cong}]$

$$\chi_x \chi_y = \sum_z g_{xy}^z \chi_z$$

$$\boxed{g_{xy}^z = \frac{|[x,z]_y|}{|\mathrm{Aut}(x)|} \prod_{i>0} \frac{|[x, z[-i]]|^{(-1)^i}}{|[x, x[-i]]|^{(-1)^i}}}$$

Here $|X| = \#(X)$

& $[x,z]_y = \{ f \in [x,z] \text{ s.t. } \exists \operatorname{cone}(f) \cong y \}$
$\subset [x,z]$

Theorem this defines an associative multiplication

- proof uses $\exists$ of dga B, (has direct proof by Xiao - Xu), statement is in terms only of triangulated category [just need T Karoubian + finiteness]

Denominator = corrected "# of automorphisms"

Numerator = corrected "# $x \to z$ with cone $\simeq y$"

<u>Remark</u> $D_{fd}(\mathcal{B})$ is the 1-truncation of an $(\infty,1)$ category $L_{fd}(\mathcal{B})$

$$\pi_i\left(L_{fd}(\mathcal{B})(x,y)\right) \simeq [x, y[-i]]$$

$L_{fd}(\mathcal{B})(x,y)$ is a finite (truncated) space or ∞-groupoid

Counting points in a finite ∞-groupoid:

$$\#(\infty\text{-groupoid } \mathcal{G}) = \sum_{f \in \pi_0(\mathcal{G})} \prod_{i>0} |\pi_i(\mathcal{G}, f)|^{(-1)^i}$$

... counting points & dividing by all higher automorphisms.

<u>Idea of proof</u>

Define space $X(B) = \mathrm{Nerve}\begin{pmatrix} \text{fd } B\text{-dg modules} \\ \text{with morphisms} = \\ \text{quasi isomorphisms} \end{pmatrix}$

$X(B)$ is a model for the ∞-category of invertible morphisms $L_{fd}(B)^{iso}$

$$\left.\begin{aligned} &\pi_0(X(B)) \simeq T/\mathrm{iso} \\ &\pi_1(X(B), x) \simeq \mathrm{aut}_T(x) \qquad x \in T \\ &\pi_i(X(B), x) = [x, x[1-i]] \end{aligned}\right\} \text{Dwyer-Kan}$$

$X(B)$ = "classifying space of fd B dg-modules"

$X^{(1)}(B)$ = " " of morphisms ($\Leftrightarrow$ monomorphisms)

$$= \mathrm{Nerve}\left\{\begin{aligned} &\text{objects: } E \hookrightarrow F \text{ monomorphism of fd } B\text{-modules} \\ &\text{morphisms: } \begin{tikzcd} E \arrow[r, hook] \arrow[d, "qis"] & F \arrow[d, "qis"] \\ E' \arrow[r, hook] & F' \end{tikzcd} \text{ strictly commutative} \end{aligned}\right.$$

$$\begin{tikzcd}
(X \hookrightarrow Z) \arrow[r, mapsto] & Z
\end{tikzcd}$$

$$\begin{tikzcd}
X^{(1)}(B) \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & X(B) \\
X(B) \times X(B) &
\end{tikzcd}$$

$(X \times Z/X)$

correspondence in spaces

π is proper, ie homotopy fibers are finite homotopy types

$\Rightarrow$ can take functions with finite support & π^* preserves that:

$Q_c(X(B))$ = locally constant functions with finite support on $\pi_0(X(B))$

$$\begin{tikzcd}
Q_c(X(B)) \otimes Q_c(X(B)) \arrow[r, "\pi^*"] \arrow[dr] & Q_c(X^{(1)}(B)) \arrow[d, "\iota_!"] \\
& Q_c(X(B))
\end{tikzcd}$$

$f: X \to Y$ map of locally finite homotopy types

$\alpha \in Q_c(X)$

$f_!(\alpha) \in Q_c(Y)$ sends

$$(y \in Y) \longmapsto \sum_{x \in \pi_0(F_y)} \alpha(j(x)) \cdot \prod_{i>0} |\pi_i(F_y, x)|^{(-1)^i}$$

homotopy fiber

$$\begin{tikzcd} F_y \arrow[r] \arrow[d, "j"'] \arrow[dr, phantom, "\lrcorner", very near start] & y \arrow[d, hook] \\ X \arrow[r] & Y \end{tikzcd}$$

. This is (!) multiplication satisfying base change!

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X' \arrow[r, "f'"] \arrow[u, "g'"] & Y' \arrow[u, "g"'] \end{tikzcd}$$

g proper

$$g^* f_! = f'_! (g')^*$$

$$\begin{tikzcd} c_! \pi^* : Q_c(X(B)) \otimes Q_c(X(R)) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & Q_c(X(B)) \arrow[d, phantom, "\cong" rotate=90] \\ H(T) \otimes H(T) \arrow[r, "\text{Hall mult.}"] & H(T) \end{tikzcd}$$

$$x \hookrightarrow y \hookrightarrow z$$

Associativity

$$\begin{tikzcd}
X^{(2)}(B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X^{(1)}(B) \arrow[r] \arrow[d] & X(B) \\
X(B) \times X^{(1)}(B) \arrow[r] \arrow[d] & X(B) \times X(B) & \\
X(B) \times X(B) \times X(B) & &
\end{tikzcd}$$

$x, y/x, z/x$

Calculate via base change: key point is homotopy Cartesian property — composing both ways just goes through same $X^{(2)}(B)$

Geometric version

B/k k commutative ring, B flat $/k$, of finite type

$$\mathcal{M}_B : \underset{\text{comm. } k\text{-algebras}}{k\text{-CAlg}} \longrightarrow \underset{\text{simplicial sets}}{\mathrm{SSet}}$$

$$k' \longmapsto X(B \otimes_k k')$$

$$X(B \otimes_k k') = \text{Nerve} \begin{cases} \text{ob: cofibrant perfect}/k' \ B \otimes_k k'\text{-modules} \\ \text{mor: quasi-isom} \end{cases}$$

cofibrant in particular $\Rightarrow$ flat $/k'$:

[model structure: fibrations = epimorphisms
equiv = quasi isom
cofibrant objects are quasifree modules]

+ base change condition
for $k' \to k''$ $\quad X(B \otimes_k k') \xrightarrow{\otimes_{k'} k''} X(B \otimes_k k'')$

$\mathcal{M}_B \in \mathrm{Ho}(\text{simplicial presheaves over } \mathrm{Aff}_k)$
$= \mathrm{Ho}(k\text{-CAG} \to \mathrm{SSet})$

- $\mathcal{M}_B$ is a stack: $\forall U_\bullet \to X$ étale (or ffqc) hypercovering of affine schemes

$$\mathcal{M}_B(X) \xrightarrow{\sim} \operatorname{holim}_{m \in \Delta} \mathcal{M}_B(U_m)$$

Theorem (Toën-Vaquié)

$\mathcal{M}_B$ is a locally algebraic stack, ie it is a union of (Zariski) open substacks which are algebraic n-stacks for varying n, of finite presentation $/k$

Alg. n-stack: $F \in Ho(SPr(Aff/_k))$ is an algebraic n-stack if $\exists$ a scheme X & a morphism $X \twoheadrightarrow F$ s.t. "fibers are smooth n-1 stacks": ie $\forall$ scheme $Y \to F$

$$\begin{tikzcd} X \arrow[r] & F \\ X \times^h Y \arrow[u] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

then $X \times^h Y$ is a smooth n-1 stack smooth over Y

$F \to Y$ is smooth if $\exists$ cover $X \to F$ s.t. $X \to Y$ is smooth

$n=0$: schemes or alg spaces, define the rest by induction

Idea of proof: Look at forgetful map $\mathcal{M}_B \longrightarrow \mathcal{M}_k$ forget k-action

Show by hand $\mathcal{M}_k$ is locally algebraic:

$\mathcal{M}_k = \bigcup_{v} \mathcal{M}_k^{v}$ perfect complexes with v giving bound on cohomological type

Cover $\mathcal{M}_k^{v}$ by strict complexes — ie matrices.

Show $\mathcal{M}_B \xrightarrow{\pi} \mathcal{M}_k$ representable

(will then imply $\mathcal{M}_B$ algebraic by a general argument: base algebraic + fibers algebraic $\Rightarrow$ total algebraic)

Representable:

$$\begin{tikzcd}
X = \operatorname{Spec} A \arrow[r, "E"] & \mathcal{M}_k \\
F \arrow[u] \arrow[r] \arrow[ur, phantom, "\llcorner"] & \mathcal{M}_B \arrow[u]
\end{tikzcd}$$

$E \longleftrightarrow$ perfect complex

$F =$ B-module structures on E

$= Map(B, End\, E)$: constructed

as iterated fiber products of $Map(B', End\, E)$

with B' free $\Rightarrow$ linear stack!

Now need to develop a good theory
of l-adic sheaves on higher stacks ....
would enable them to develop geometric Hall algebra

$H^{geom}(B) = K_0(D(M_B))$

Difficulty: defining pushforward with proper support
(Laszlo-Olsson) - not bounded but weights
increase so trace functions well defined.

$\Rightarrow$ Get associative product on $D(\mathcal{M}_B)$

$k = \mathbb{F}_q \Rightarrow H^{geom}(B) \xrightarrow{Tr} H(T^!$

$E \in D(\mathcal{M}_B) \longmapsto (x \longmapsto Tr_{Fr_x} \bar{E}_x)$

Use Trace Formula - to know pushforward commutes with trace.

Absolute Hall algebra:

$H^{abs}(B) = K_0(\text{alg stacks}/\mathcal{M}_B)$

$K_0(\text{alg stacks}/k)$: use only special alg stacks

F is special if $\forall x : \operatorname{Spec} K \to F$ field pnt

$\pi_1(F, x)$ is represented by a linear alg group $/k$

& $\pi_1(F, x)$ is rep. by unipotent alg group $/k$

... e.g. $\mathcal{M}_B$ (or more precisely its finite type substacks)

$$K_0(\text{alg stacks}/k) = \frac{\mathbb{Z}[\text{special alg stacks}]}{\begin{array}{ll} \bullet\ F_0 \underset{\text{closed}}{\hookrightarrow} F & \bullet\ F \to F' \text{ a Zariski local fibration with fiber } F_0 \\ [F] = [F_0] + [F \setminus F_0] & [F] = [F' \times F_0] \end{array}}$$

Theorem

$$K_0(\mathrm{Var})[L^{-1}, (L^{i}-1)^{-1}_{i>0}] \xrightarrow{\ \sim\ } K_0(\text{alg stacks})[L^{-1}, (L^{i}-1)^{-1}_{i>0}]$$

ie invert all $[\mathbb{A}^i - 0]$

Corollary $\exists\ h^2_{xy} \in K_0(\mathrm{Var})[L^{-1}, (L^{i}-1)^{-1}_{i>0}]$ s.t

$(k = \mathbb{F}_q)$ $\cancel{\#}\ h^2_{xy}(\mathbb{F}_q) = S^2_{xy}$

e.g $G \hookrightarrow GL_n$ $\quad [BG] = \dfrac{[GL_n/G]}{[GL_n]}$

& $\dfrac{1}{[GL_n]}$ is in our localized ring

<u>Corollary</u> $g_{xy}^{z} \in \mathbb{Z}[q^{-1}, (q^{i}-1)^{-1}]$

- Hall numbers are incarnations of <u>motivic</u> Hall numbers.