V. Toledano-Laredo : Stokes Matrices & Stability Conditions

Note Title 2/6/2008

(w/ Tom Bridgeland)

Stokes interpretation of Joyce's generating function for counting invariants in abelian categories
($\mathcal{A}$ = Mod R f.d. algebra)

- Review of Stokes data
- computation via multilogs
- application to Joyce's work

G affine alg. group / $\mathbb{C}$ $\supset H$ torus
$\mathfrak{g} \supset \mathfrak{h}$ Lie algebras
$\mathfrak{g} = \mathfrak{g}_0 \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ $\Phi \subset \mathfrak{h}^*$ "root" system
(G not nec. semisimple)

Examples:
I. G semisimple $\supset H$ max torus
II. $G = GL(n)$ $H \supseteq (\mathbb{C}^*)^m$ non maximal torus
III. $G = N \rtimes H$ unipotent with torus action

e.g. $\hat{N}(\mathcal{A}) \supset H = \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{A}), \mathbb{C}^*)$
$\mathfrak{h} = \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{A}), \mathbb{C}) \supset \mathrm{Stab}\, \mathcal{A}$

P trivial principal G-bundle on $\mathbb{P}^1$

$$\nabla = d - \left(\frac{Z}{t^2} + \frac{f}{t}\right)dt \qquad Z, f \in \mathfrak{g}$$

$$= d + \left(Z + \frac{f}{\tau}\right)d\tau \qquad \tau = t^{-1}$$

Assumptions : I. $Z \in \mathfrak{h}$ ($\Rightarrow Z$ semisimple
$\Rightarrow \mathfrak{g} = \mathfrak{g}^Z \oplus [Z, \mathfrak{g}]$)

II. $\pi_{\mathfrak{g}^Z}(f) = 0$ ie $f \in [Z, \mathfrak{g}]$

Aim Construct solutions of ∇ having good asymptotic properties, $\Phi \sim e^{-Z/t}$

ie $\Phi = F \cdot e^{-Z/t}$ where $F = F(t)$ holomorphic on $\mathbb{C}$, $F(0) = 1$

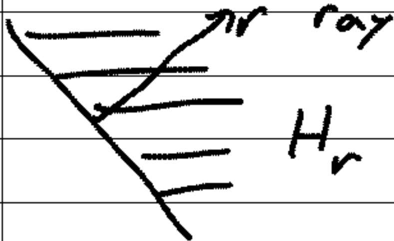
Problem: can only find such F formally

[Exercise: $\nabla = d - \left(\frac{\begin{pmatrix} a & \\ & b \end{pmatrix}}{t^2} + \frac{\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}}{t}\right) dt$]

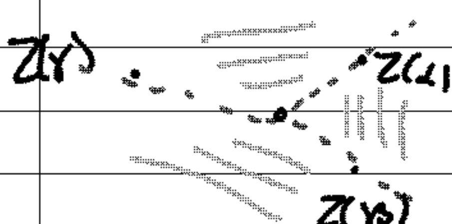
[Note if f had nontrivial diagonal part look for $\Phi = F\, e^{-Z/t}\, t^{\pi_{\mathfrak{g}^Z}(f)}$:

formally $\nabla \sim d - \frac{Z}{t^2} + \frac{\pi_{st}}{t}\, dt$]

Next best: look for solutions $\Phi \sim e^{-\frac{Z}{t}}$ on halfplanes:



Def A Stokes ray of ∇ is a ray of the form
$\mathbb{R}_{>0} \cdot Z(\alpha) \qquad \alpha \in \Phi$
... $Z(\alpha)$ = nonzero eigenvalues of Z on $\mathfrak{g}_\alpha$



Classically: Z diagonal

$Z = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}$ &

Stokes rays are $\mathbb{R}_{>0}(z_i - z_j)$

For simplicity in this talk assume Z is regular
$Z(\alpha) \neq 0 \quad \forall \alpha$

Stokes sectors: connected components of $\mathbb{C}^* \setminus$ rays

(Sibuya, Balser-Jurkat-Lutz, Boalch, B-TL)

<u>Theorem</u> If r is not a Stokes ray

$\exists!$ fundamental solution of ∇ ϕ_r

- holomorphic on $\mathbb{H}_r$
- $\phi e^{Z/t} \longrightarrow 1$ as $t \longrightarrow 0$ in $\mathbb{H}_r$

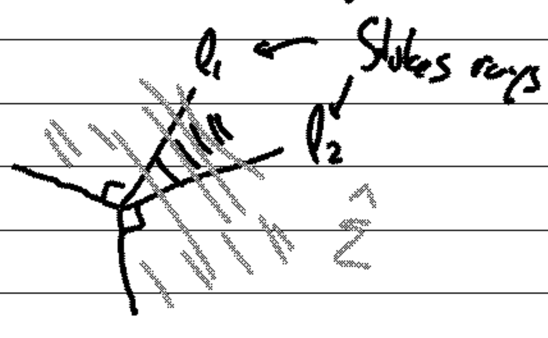
Comparing rays:  $\Sigma = \mathbb{H}_{r_1} \cap \mathbb{H}_{r_2}$

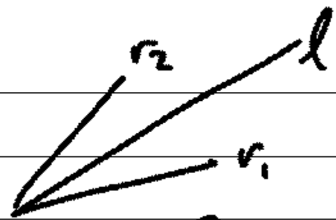
$\phi_{r_1} = \phi_{r_2} \cdot S_\Sigma$, $S_\Sigma \in G$

<u>Prop</u> S_Σ is unipotent & $\log S_\Sigma \in \bigoplus \mathfrak{g}_\alpha$
$Z(\alpha) \in \overline{\Sigma} \setminus \{0\}$

<u>Cor</u> If Σ does not contain any Stokes rays
$\Longrightarrow S_\Sigma = 1 \Longrightarrow \phi_r$ extends to a solution of ∇ having required asymptotics on the <u>Stokes supersector</u> $\hat{\Sigma} = \bigcup_{r \in \Sigma} \mathbb{H}_r$

ie





Stokes ray

$$\Rightarrow \phi_{r_1} = \phi_{r_2} \cdot S_\ell \, ,$$

$S_\ell \in G$ indep of r_1, r_2

$$\log S_\ell \in \bigoplus_{Z(\alpha)\in \ell} \mathfrak{g}_\alpha$$

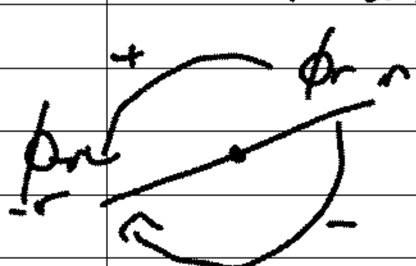
<u>Stokes map</u> $\quad S_\ell = \exp\Big(\sum_{Z(\alpha)\in\ell} \varepsilon_\alpha^{\in \mathfrak{g}_\alpha} \Big)$

$$\begin{aligned} S : [Z, \mathfrak{g}] &\longrightarrow [Z, \mathfrak{g}] \\ f &\longmapsto \varepsilon = \sum \varepsilon_\alpha \end{aligned}$$

Boalch: G reductive $\Rightarrow S$ is a local analytic isomorphism.

<u>Stokes multipliers</u>

r a ray s.t. $r, -r$ <u>not</u> Stokes rays



$$\phi_{-r} \cdot S_{\pm} = \phi_r$$

$S_\pm \in G$ Stokes multipliers of r

Stokes factors vs Stokes multipliers:

$$S_+ = \overrightarrow{\prod_{\ell \subset \ldots r}} S_\ell \qquad S_- = \overrightarrow{\prod_{\ell \subset \ldots r}} S_\ell^{-1}$$

.... reminiscent of Reineke's Harder-Narasimhan product $\quad \mathcal{I}_\ell = \overrightarrow{\prod} SS_\ell \quad$ semisimple objects with $Z([A]) \in \ell$

Theorem (BTL)

1) The Stokes map S is given by a Lie series of the form

$$\varepsilon_\alpha = \sum_{n \geq 1} \sum_{\alpha_1 + \ldots + \alpha_n = \alpha} L_n(Z(\alpha_1), \ldots Z(\alpha_n)) f_{\alpha_1} \cdots f_{\alpha_n}$$

which is absolutely convergent in Λ, L

$$L_n(z_1, \ldots z_n) = \int_{[0, s_n]} \frac{dt}{t - s_1} \circ \ldots \circ \frac{dt}{t - s_{n-1}}$$

$s_i = z_1 + \ldots + z_i$ (provided $s_i \notin]0, s_n[$ $\quad i = 1 \ldots n-1$)

(multivalued fn...)

Lie series: only involves commutators..

11) The Taylor series at $\varepsilon = 0$ of the inverse of ς is given by a Lie series of the form

(*) $$f_\alpha = \sum_{n \geq 1} \sum_{\alpha_1 + \dots + \alpha_n = \alpha} J_n(Z(\alpha_1) \dots Z(\alpha_n))\, \varepsilon_{\alpha_1} \dots \varepsilon_{\alpha_n}$$

where J_n is holomorphic on $\{ \underline{z} \in (\mathbb{C}^*)^n : \frac{z_{i+1}}{z_i} \notin \mathbb{R}_{>0} \}$

& coincide with Joyce's generating functions F_n.

Relation to Joyce's Work

$\mathcal{A} = \mathrm{Mod}(R)$ $\quad R$ fd algebra / $\mathbb{C}$

$\mathcal{H}(\mathcal{A})$ = Hall algebra = constructible functions on

$$\mathcal{M}(\mathcal{A}) = \coprod_{d \geq 0} \mathrm{Rep}_d(R) / GL_d$$

$\cup$

$C(\mathcal{A})$ = algebra generated by χ_α = char. function of objects of class $\alpha \in K_0(\mathcal{A})$

$Z \in \mathrm{Stab}\,\mathcal{A} : \{ Z \in \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{A}), \mathbb{C}) : Z(K_{>0}) \subset \mathbb{H} \}$

SS_ℓ = char fn. of Z-semistable objects of class γ
s.t. $Z(\alpha) \in \ell$

$\widehat{C(\mathcal{A})}$

$1_{\mathcal{A}}$ = char fn of all objects

1_0 = char fn. of 0 object.

Reineke: $1_{\mathcal{A}} = \overrightarrow{\prod_{\ell \in \mathbb{H}}} SS_\ell$

$\widehat{N(\mathcal{A})}$: grouplike elements

$\widehat{B(\mathcal{A})} = \hat{N}(\mathcal{A}) \rtimes H$ $\qquad H = \mathrm{Hom}(K_0(\mathcal{A}), \mathbb{C}^*)$

$\hat{P}$ = principal $\hat{B}(\mathcal{A})$ bundle on $\mathbb{P}^1$

$$\nabla_{\mathcal{A},Z} = d - \left(\frac{Z}{t^2} + \frac{f}{t}\right) dt$$

$Z \in \mathrm{Stab}\,\mathcal{A} \subset \mathfrak{h}$ $\quad \Delta$

$f \in \hat{\mathfrak{n}}(\mathcal{A})$

Can construct ϕ_r fundamental solutions as above ---
but note there can be ∞ many Stokes rays!

Can still define Stokes multipliers $S_\pm$ & factors
(can still define in the pro setting here
as limits of fd versions)
.. here $S_\pm = S_\pm(\mathbb{R}_{>0})$: use
basic ray $\mathbb{R}_{>0}$.

$S_+ = \overrightarrow{\prod} S_\ell$: infinite product but
well defined in projective limit.

Note: our root system only has positive roots
$Z(\alpha) \in \mathbb{H}$!

<u>Corollary</u> Define f by $(*)$ above
where $\varepsilon_\alpha = \varepsilon_\alpha^{s}$ defined last time
(modifications of char. functions of semistable
objects of class α :
-- ie $\log SS_\ell =: \sum_{Z(\alpha) \in \ell} \varepsilon_\alpha^{ss}$)

1) The Stokes factor S_ℓ of $\nabla_{f,Z}$ is SS_ℓ

2) The Stokes multipliers are $S_+ = 1_{\mathcal{A}}$
$S_- = 1_{\mathcal{O}}$

[no Stokes rays in lower half plane!]

[$\boxed{\text{Isomonodromy}}$ [back to general setting

Family of connections $\nabla = d - \left(\frac{Z}{t^2} + \frac{f(Z)}{t}\right)dt$

is <u>isomonodromic</u> if $\exists$ flat holomorphic total connection $\overline{\nabla}$ on $\mathbb{P}^1 \times (U \subset \mathfrak{h})$ (small patch)

s.t. $\overline{\nabla}|_{\mathbb{P}^1 \times \{Z\}} = \nabla$ [& require pole of order at most 2 on $t=0$]

<u>Theorem</u> (Jimbo-Miwa-Ueno $G = GL_n$,
Boalch G reductive)

TFAE : I. ∇ is an isomonodromic family

II. Up to shrinking U $\exists$ a ray r st. $\pm r$ not Stokes rays for $\overline{\nabla}|_{\mathbb{P}^1 \times \{Z\}}$

& the corresponding Stokes multipliers $S_\pm = S_\pm(r, Z)$ are constant in Z.

III $$df_\alpha = \sum_{\beta+\gamma=\alpha} [f_\beta, f_\gamma]\, d\log\left(\frac{\beta}{\gamma}\right)$$

"Schlessinger" equation for second order pole.

Theorem (BTL)

i) The $f_\alpha(Z)$ defined by *
depends holomorphically on Z

ii) $df_\alpha = \sum_{\beta+\gamma} [f_\beta, f_\gamma] \, d\log \frac{\beta}{\gamma}$

..... the Stokes multipliers $S_+ = 1_\mathcal{A}$, $S_- = 1_\circ$
are independent of Z!

[$\nabla_{\mathcal{A},Z}$ is completely determined by
Z, $S_+ = 1_\mathcal{A}$, $S_- = 1_\circ$!]