Y. Varshavsky: Towards a global Langlands correspondence for function fields

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joint w/ Kazhdan

I. Formulation

X smooth proj. (geom. irreducible) curve $/\mathbb{F}_q$

$K = \mathbb{F}_q(X)$ $\quad \mathbb{A} = \mathbb{A}_F$ adèles

$\Gamma_K = \mathrm{Gal}(\overline{K}/K)$ $\quad = \bigotimes'_{v \in X} K_v$

ℓ prime $\quad G$ split reductive group $/\mathbb{F}_q$

$\hat{G} = {}^L G^0$ is Langlands dual group $/\overline{\mathbb{Q}_\ell}$

Langlands conjecture: For $\forall$ cuspidal automorphic rep π of $G(\mathbb{A})$ with central character

$$Z_\pi : Z(G(\mathbb{A})) \longrightarrow \overline{\mathbb{Q}}_\ell^* \text{ of finite order}$$

$\exists$ a continuous homomorphism $\rho_\pi : \Gamma_K \to \hat{G}(\overline{\mathbb{Q}}_\ell)$ unramified almost everywhere, s.t.

$$[\rho_\pi(Fr_v)]^{ss} = c(\pi_v) \quad \text{for almost all } v$$

conjecturally the $[\,]^{ss}$ is unnecessary

Explanations: Automorphic rep ~ subquotient of functions on $G(K) \backslash G(\mathbb{A})$, irreducible admissible.

Any irred admissible π of $G(\mathbb{A})$ can be written
$$\pi = \bigotimes_{v \in X}' \pi_v \quad \text{with } \pi_v \text{ irreducible smooth}$$
rep of $G(K_v)$, unramified almost all v.

Satake isomorphism:
$$\pi_v \in \mathrm{Irr\,Unramified}(G(F_v)) \longleftrightarrow \hat{G}^{ss}(\overline{\mathbb{Q}}_\ell)/\text{conjugation}$$
$$\text{is} \qquad\qquad\qquad\qquad \ni$$
$$\chi : \mathcal{H}(G(F_v), G(\mathcal{O}_v)) \longrightarrow \overline{\mathbb{Q}_\ell} \qquad c(\pi_v) = c_v(\pi)$$
$$\updownarrow$$
$$\overline{\mathbb{Q}}_\ell[\hat{T}]^W$$

<u>Remark</u> $\forall \pi$ $\exists$ automorphic character
$\chi : G(\mathbb{A}) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ s.t. $Z_{\pi \otimes \chi}$ is of finite order - so can guarantee finite order by twist.

Conjecturally, every π gives rise to
$$\rho_\pi : W_F \longrightarrow \hat{G}(\overline{\mathbb{Q}}_\ell) \quad \text{Weil group rep.}$$

<u>Corollary of conjecture</u> For all π of above and $\forall\, \omega \in \operatorname{Rep} \hat{G}$

$\exists$ Galois representation $\rho_{\pi,\omega}\ (= \omega \circ \rho_\pi)$ of Γ_K s.t.

$$(*) \quad \operatorname{Tr}\left(\rho_{\pi,\omega}(Fr_v)^m\right) = \operatorname{Tr}\left(\omega \circ c_v(\pi)^m\right)$$

$\forall m \geq 0$, almost all v.

[$\rho : \Gamma_K \longrightarrow \hat{G}(\overline{\mathbb{Q}}_\ell)$ is <u>unramified</u> at v if inertia $I_v \subset \operatorname{Ker} \rho$, $\Rightarrow$ $\rho(Fr_v)$ defined up to conjugacy]

[$\rho_{\pi,\omega}$ unique but ρ_π not unique ... ie commutativity constraint not unique!]

GL_1 : class field theory
GL_2 : Drinfeld
GL_r : Lafforgue

First main theorem Let G be split reductive $/\mathbb{F}_q$ & π as before, s.t. π_v belongs to a cuspidal L-packet for some $v \in X$.
Then $\exists$ a virtual Galois representation $\rho_{\pi,w} \in \mathbb{Z}[\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell} \Gamma_K]$ satisfying $*$.

Definition By cuspidal L-packet of $G(K_v)$ we mean a minimal finite set $\Pi = \{\rho_1, \ldots \rho_r\}$ of smooth irreducible reps of $G(K_v)$ s.t.

1 • ρ_i are all cuspidal
2 • ρ_i is induced from a compact (mod center) subgroup (up to multiplicity)
3 • $\sum l_i \chi(\rho_i)\big|_{G^{rss}(F_v)}$ is stable $(l_i \in \mathbb{N})$

— i.e. know how to stabilize all ρ_i by a collection of cuspidals

Condition 2 almost known to follow from 1: most cuspidals are known to be compactly induced.

Example Every cuspidal Deligne-Lusztig rep. belongs to a cuspidal L-packet at least if char $\mathbb{F}_q$ is good (>5 suffices!)

More precisely let $a: T \hookrightarrow G$ be an embedding of a max. elliptic torus / F_v which is split over F_v^{unr} max unramified extension.

& let $\theta: T(K_v) \longrightarrow \overline{\mathbb{Q}}_\ell^*$ in general position + tamely ramified

$\rightsquigarrow$ cuspidal L-packet!

let $a_1, \dots, a_\ell$ be the set of embeddings $a_i : T \hookrightarrow G$ up to conjugacy s.t.

$a_i \sim a$ stably.

Then $a_i \rightsquigarrow \pi_{a_i, \theta}$ cuspidal irreps of $G(K_v)$

(induction of inflation of usual Deligne-Lusztig rep — so by definition is induction from a parahoric.)

Theorem : $\sum \chi(\pi_{a_i,\theta})|_{G^{rs}(F_v)}$ is stable

<u>Goal</u> : Construct a scheme $\mathfrak{X}_{\bar{w}}/k \circlearrowleft G(\mathbb{A})$

s.t. $\rho_{\pi,w}$ is a subquotient of $H_c^*(\mathfrak{X}_{\bar{w}}, IC)$

II. Moduli of F-bundles

Fix an n-tuple $\bar{w} = (w_1, \dots w_n) \in \mathrm{Irr}(\hat{G})^n$

s.t. $\otimes(w_i|_{Z(\hat{G})}) = 1$

$\rightsquigarrow$ Deligne-Mumford stack $F\text{-}Bun_{n,\bar{w}} / X^n$

informally : $F\text{Bun} = (\mathcal{G}, \bar{x}, \varphi)$ s.t

$\mathcal{G}$ is a G-bundle on X.

$\bar{x} = (x_1, \dots x_n) \in X^n$

$\varphi : {}^{\tau}\mathcal{G}|_{X \setminus \{x_1 \dots x_n\}} \simeq \mathcal{G}|_{X \setminus \{x_1 \dots x_n\}}$

where ${}^{\tau}\mathcal{G} = \phi_X^* \mathcal{G}$, ϕ_X = arithmetic Frobenius,

& pole of φ at x_i are $\leq w_i$

[ie singularity is closure of Schubert cell in affine Gr]

Ex 1: $\omega_i = 1 \ \forall i \Rightarrow {}^{\tau}\mathfrak{g} \simeq \mathfrak{g}$

so $FBun_{n,\omega}$ is almost iso (& exactly for GL_n) a discrete set of pts (stack)

$\left(G(K) \backslash G(\mathbb{A}) / G(\mathbb{O}) \right) \times X^n$

.... not true in general since not every bundle trivializes on a Zariski open: in general need some inner forms above, union of such.

Ex 2 $\quad \mathfrak{g} = GL_1, \quad \operatorname{Irr} \hat{G} = \mathbb{Z}$

$\bar{\omega} = (k_1 \ldots k_n)$ with $\sum k_i = 0$

$$\Rightarrow \quad \begin{tikzcd} FBun_{n,\bar{\omega}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Pic^0 X \arrow[d, "a"] \\ X^n \arrow[r, "b"'] & Pic^0 X \end{tikzcd}$$

with a = Lang isogeny

$b(x_1, \ldots x_n) = \mathcal{O}(\sum k_i x_i)$

Ex 3 $G = GL_n$ $W_1 = \text{standard}$, $W_2 = \text{standard}^*$

$\Rightarrow FBun_{n,w} = F$-sheaves, considered by Drinfeld & Lafforgue.

$\forall D \subset X$ finite $\rightsquigarrow FBun_{n,w,D}$:

triples with trivialization over D.

- stack over $(X \setminus D)^n$, Galois cover of

$FBun_{n,w}|_{(X\setminus D)^n}$ with Galois group $G(\mathcal{O}_D)$.

Definition $\mathcal{X}_{\bar{w}} = \varprojlim_D FBun_{n,\bar{w},D}$ [up to completing the center]

a scheme over $\operatorname{Spec} \mathbb{F}_q(X^n)$,

with $G(\mathbb{A})$ action.

Notation Set $V_{\bar{w}} = \bigoplus_i (-1)^i V_{\bar{w}}^i := \bigoplus_i (-1)^i H_c^i(\mathcal{X}_{\bar{w}}, IC)$

virtual representation.

Note: IC of $\mathcal{X}_{\bar{w}}$ makes sense since

$\mathcal{X}_{\bar{w}} = \varinjlim_{\text{open}} \varprojlim_{\text{étale}}$ (quasiproj. schemes)

Singularities are those of strata in affine Grassmannian
locally isomorphic to a Grassmannian
— same as argument that Deligne-Lusztig varieties have same singularities as Schubert cells.

$V_{\bar{w}}$ is a smooth virtual rep of $G(\mathbb{A}) \times \Gamma_{\pi_1^{\xi}(X^n)}$

<u>NB</u>: $V_{\bar{w}}$ is not admissible!
FBun only locally of finite type, but not of finite type.

Example! $w_i = 1 \ \forall i \Rightarrow X_{\bar{w}} \simeq G(k) \backslash G(\mathbb{A})^{\wedge}$ ← complete conts
so $V_{\bar{w}}^{*} = \mathrm{Fun}(G(k) \backslash G(\mathbb{A}))$
Space of automorphic forms.

III. Explicit form of First Main Theorem.

Assume G is semisimple & simply connected

Then $w \in \mathrm{Irr}\, \hat{G} \rightsquigarrow V_w \circlearrowleft G(\mathbb{A}) \times \Gamma_k$

Fix cuspidal aut. rep $\pi = \pi_v \otimes \pi^v$

where $\pi_v \in \Pi$, a cuspidal L-packet
$\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{\{\rho_1 \ldots \rho_k\}}$

<u>Definition</u> Set $m_{cusp}(\Pi \otimes \pi^\vee) = \sum m_{cusp}(\rho_i \otimes \pi^\vee)$

cuspidal multiplicities:

$$m_{cusp}(\rho_i \otimes \pi^\vee) = \dim \operatorname{Hom}_{G(\mathbb{A})}(\rho_i \otimes \pi^\vee, C_{usp}(G(F) \backslash G(\mathbb{A})))$$

Set $(V_w)_{\Pi \otimes \pi^\vee} = \bigoplus (V_w)_{\rho_i \otimes \pi^\vee}$

$$(V_w)_{\rho_i \otimes \pi^\vee} = \operatorname{Hom}_{G(\mathbb{A})}(\rho_i \otimes \pi^\vee, V_w^{ss})$$

treat as virtual rep: ignores all non-semisimplicity.

Set $\rho_{\Pi, w} =$

$$\frac{1}{m_{cusp}(\Pi \otimes \pi^\vee)} (V_w)_{\Pi \otimes \pi^\vee} \in \mathbb{Q}[\operatorname{Rep}_{\mathbb{Q}_\ell} \Gamma_E]$$

- virtual rep with rational coefficients.

<u>Second Main Theorem</u> $\rho_{\pi,w} \in \mathbb{Z}[\mathrm{Rep}_{\mathbb{Q}_\ell}\Gamma_k]$
and satisfies $*$.

<u>Remark</u> This theorem implies that

$$\bigoplus_{\rho_i \in \pi} (V_w)_{\rho_i \otimes \pi^\vee} = \bigoplus_{\rho_i \in \pi} m_{cusp}(\rho_i \otimes \pi^\vee)\, \rho_{\pi,w}$$

<u>But</u> this is false without sum,
even for $G = SL_2$: it can happen
that $m_{cusp}(\rho_i \otimes \pi^\vee) \neq 0$ but
$(V_w)_{\rho_i \otimes \pi^\vee} = 0$ .... ie endoscopic groups appear!

<u>Conjecture</u> 1) $(V_w)_\pi$ is pure of weight;
($\Longrightarrow$ $(V_w^i)_\pi = 0$ all i odd $\&$
our virtual rep is an actual rep)
[would follow from having nice compactification]

2) $\exists \rho_\pi : \Gamma_k \longrightarrow \hat{G}(\overline{\mathbb{Q}}_\ell)$ st $\rho_{\pi,w} = w \circ \rho_\pi$

IV Strategy of the proof

Property * almost implies integrality!

Study *: assertion on traces, but V_w not admissible....

Define $(V_w)_\pi$ isotypic component, virtual rep of $G(\mathbb{A}^v) \times \Gamma_K$.

Third Main Theorem a) $(V_w)_\pi$ is admissible

b) $Tr\left(h^{u,v} \otimes \underbrace{1_u \times Fr_u^m}, (V_w)_\pi\right)$

[in Hecke algebra of $\mathbb{A}^{v,u}$ adeles off two places u, v]

$= Tr\ h^{u,v} \otimes h_u(m,w) \otimes h_v, \left(\operatorname{Cusp}_{G(F)}^{G(\mathbb{A})}\right)$

where $h_u(m,w) \in \mathcal{H}(G(K_u), G(\mathcal{O}_u))$ unramified Hecke

$$\begin{array}{ccc} h_u(m,w) & & \mathcal{H}(G(K_u), G(\mathcal{O}_u)) \\ \updownarrow & & \wr\!\!\mid \\ g \mapsto Tr(w(g^m)) & \in & \overline{\mathbb{Q}}_\ell\,[\hat{T}/W] \end{array}$$

base change

and h_v is an explicit linear combination of matrix coefficients of ρ_i's ... compactly supported! (induced from compact subgroup)

Π is stable $\Rightarrow$ h_v is stable.

Idea of proof : $(V_w^i)_{\rho_i} = H_c^i(\mathfrak{X}_{\bar{w}}/H_i, \mathcal{I}_i)$

for certain irreducible perverse sheaves.

If $\rho_i = \mathrm{Ind}_{H_i}^{G(K_v)}(\tau_i)$ τ_i rep on W_i finite dim

$$\mathcal{L}_i = \left(IC_{\mathfrak{X}_{w_i}} \otimes_{\overline{\mathbb{Q}}_\ell} W_i \right) / H_i$$

IC sheaf descended to $\mathfrak{X}_{\bar{w}}/H_i$ from W_i

<u>Theorem</u> $\mathcal{L}_i$ is supported on an open subset of pro finite type (analog of <u>compact support</u>).

Now use generalized Fujiwara Lefschetz trace formula & simple Arthur trace formula + fundamental lemma for stable base change.

Fundamental lemma for stable base change:

E/F finite unramified extension:

$N: G(E)/_{\text{stable twisted conjugacy}} \longrightarrow G(F)/_{\text{stable conjugacy}}$ norm map

Or

Stable twisted orbital integrals of a spherical function f = stable orbital integral $O_{N(\gamma)}$ for base change of f

Where base change: $\mathcal{H}(G(E), G(\mathcal{O}_E)) \to \mathcal{H}(G(F), G(\mathcal{O}_F))$

corresponds under Satake to

$f(g) \longmapsto f(g^n)$.

(Kottwitz for unit, Clozel & Labesse in general).