B. Webster - Hypertoric varieties & Koszul dualities

Note Title 4/2/2008

(with Braden, Licata, Proudfoot)

Story time

What if ... this was an audience of combinatorialists?

fin dim algebra $A(V)$

Polarized arrangement $\xrightarrow{\text{gens + rels}}$ $A(V)$

(combinatorics: hyperplane arrangement + extra data)

$V = (\{H_i\}, \chi, \xi)$

$A(V) \longrightarrow$ category $\mathcal{C}(V) = \mathrm{Rep}\, A(V)$

hypertoric variety U_χ $\longrightarrow$ Fukaya(U_χ) / microlocal perverse sheaf / D-modules

Theorem [BLPW]

1. $A(V)$ is Koszul & quasihereditary
   $\rightsquigarrow$ $\mathcal{C}(V)$ is a highest weight category
2. The center $Z(A(V))$ is the <u>reduced Stanley-Reisner ring</u> of the arrangement (or its associated matroid)
3. Different polarizations χ, ξ give algebras A which are derived Morita equivalent
   ... $D(\mathcal{C}(V))$ is an invariant of $\{H_i\}$.

(polarizations of hyperplane arrangement give t-structures on $\mathcal{C}(V)$)

4. The Koszul dual $A(V)^! = A(V^\vee)$ is associated to the Gale dual arrangement $V^\vee$.
(will become more mysterious as we progress..)

Hypertoric variety: $T \circlearrowright \mathbb{C}^I$ torus acting on a vector space.
Character χ gives a GIT stable locus
$U_\chi \subset T^*(\mathbb{C}^I/T)$

ξ gives $\mathbb{C}^*$ action $\mathbb{C}^*_\xi \circlearrowright \mathbb{C}^I/T$

We'll look at perverse sheaves on $\mathbb{C}^I/T$ which are constructible wrt the Bialynicki-Birula stratification associated to $\mathbb{C}^*_\xi$, conormals W_ξ, & modding out by those whose microsupport misses U_χ:

$$\mathrm{Perv}_V = \{F \mid \mu\mathrm{supp}(F) \subset W_\xi\} \Big/ \{F \mid \mu\mathrm{supp}(F) \cap U_\chi = \emptyset\}$$

[W_ξ : some of the subvarieties in the extremal core (includes all of core directions)]

<u>Theorem</u> [BLPW] $\mathrm{Perv}_V = \mathcal{C}(V)$

[Note U_χ is actually a quasiprojective variety - maybe orbifold points if arrangement not unimodular ..]

$Z(\mathcal{A}(V))$ will be the cohomology of the dual hypertoric variety ....

<u>Mystery</u> Why is Perv_V Koszul dual to $\mathrm{Perv}_{V^\vee}$?

eg $\mathrm{Fuk}(\widetilde{\mathbb{C}^2/\mathbb{Z}_k})$ is KD to $\mathrm{Fuk}(T^*\mathbb{P})$!

Speculation [BLPW] This is a special case of a more general duality on hyperkähler varieties, with some extra data (switching χ, ξ actually changes varieties)

$\mathcal{M} \rightsquigarrow$ "category $\mathcal{O}$" category $\mathcal{O}^{\mathcal{M}}$ (quasihereditary, ...)

& $\exists\ \mathcal{M}^\vee \rightsquigarrow \mathcal{O}^{\mathcal{M}^\vee}$ Koszul dual

- Should include Goresky-MacPherson duality
$H^2_T(M) \longrightarrow H^2_T(pt)$ gives arrangement
- subspaces of various projections to $H^2_T(M^!)$...

Known examples:

- Type A: Slodowy slices $(\lambda, \mu) \longmapsto (\mu^t, \lambda^t)$
transpose
(maybe other types? Flag varieties of Langlands dual groups are GM dual...)
$\rightsquigarrow$ BGS duality on original category $\mathcal{O}$
- singular / parabolic blocks (one of $\lambda \mu$ gives singularity, other gives parabolicity)

- $U(k)$ instantons on ALE space for A_N $\widetilde{\mathbb{C}^2/\mathbb{Z}_N}$
$\longleftrightarrow$
$U(N)$ instantons on $\widetilde{\mathbb{C}^2/\mathbb{Z}_k}$
(Licata: geometric realization of Boson-Fermion corresp; categorical side should involve Cherednik algebras)

Physics (Gaiotto-Witten)

Higgs branch of $\mathcal{N}=4$ 3d superconformal field theory $\longleftrightarrow$ Coulomb branch of same theory = Higgs branch of mirror

[GM for flag varieties : H^2's for dual flag varieties are both the torus.
For each fixed point get map $H^2_T(\text{flag}) \to H^2_T(pt)$
Matching up fixed points for dual varieties find kernels annihilate each other
-- flag varieties these kernels are graphs of Weyl group action..]

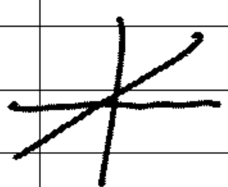
End of story-time: time for math!

Def A polarized arrangement is $V^+ = (V, \chi, \xi)$

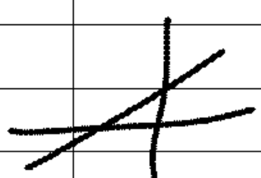
- $V \subset \mathbb{R}^I$ subspace $\quad V \xrightarrow{i} \mathbb{R}^I \xrightarrow{\pi} \mathbb{R}^I/V$
- $\chi \in \mathbb{R}^I/V \cong (V^\perp)^\wedge$ $\leftarrow$ foreshadowing
- $\xi \in V^* \cong \mathbb{R}^I/V^\perp$

with everything generic.

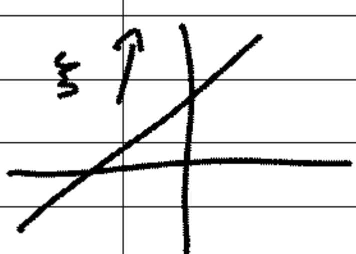
Draw picture: $\mathbb{R}^I$ has hyperplane arrangement from coordinate hyperplanes, so let's draw the intersection of V with these hyperplanes



$V \cap \{x_i = 0\}$



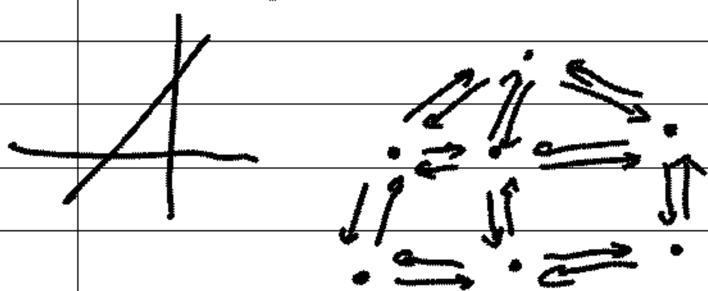
To draw χ: replace V by $V + \chi = \pi^{-1}(\chi)$



ξ is a function on $V \rightsquigarrow$ function on $V+\chi$ up to shift, gives notion of which points are "higher" than others - direction

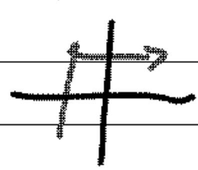
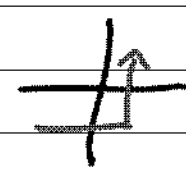
1) Chamber := component of the complement
2) A chamber is bounded if ξ achieves a maximum on closure ($\rightsquigarrow$ nec. unique by genericity)

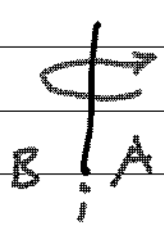
Quiver $\mathcal{V}$: vertices are chambers, edges are adjacencies of chambers



Cayley graph of the Deligne groupoid of the arrangement.

The path algebra $\Pi(\mathcal{V}) \otimes \mathrm{Sym}^\bullet(V^*)$ has a quotient $A(V)$ by the following paths in arrangement:

1.  $=$ 

2. 

$= j^*(x_i)\, e_A$

idempotent of A (lazy path)

pull back ith coordinate function in $\mathrm{Sym}^\bullet V^*$

Since x_i span V^*, get every element of $\mathrm{Sym}^\bullet V^*$

from crossing walls, but include linear relations

$\Rightarrow$ so path algebra $\Pi(V)$ will surject onto this quotient of $\Pi(V) \otimes \mathrm{Sym}^{\cdot} V^*$

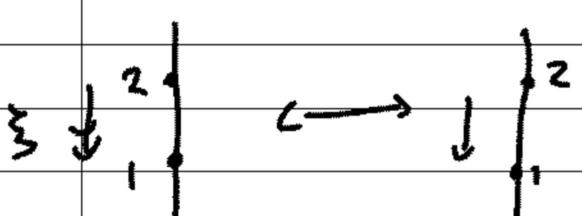
3. $e_A = 0$ if A is unbounded

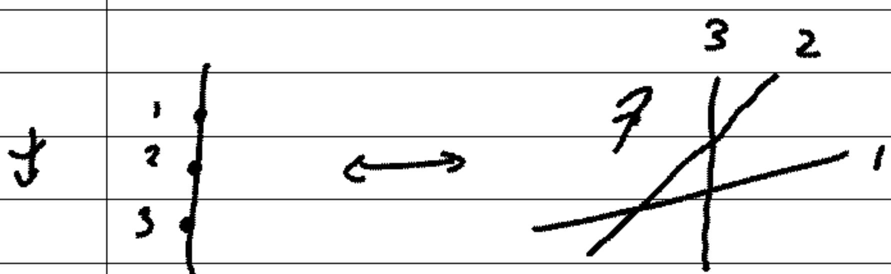
<u>Gale duality</u> $\quad \mathcal{V} = (V \subset \mathbb{R}^I, \chi, \xi) \qquad \dim V = d$

$\rightsquigarrow$ dual $\quad \mathcal{V}^\vee = (V^\perp \subset \mathbb{R}^I, -\xi, -\chi) \qquad \dim V^\vee = I - d$

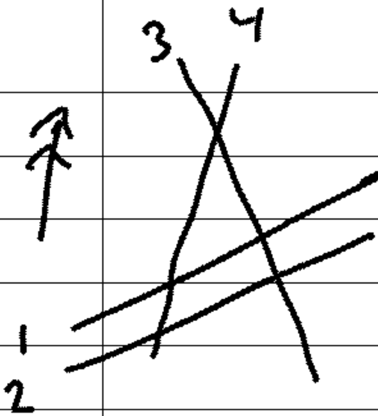
1-dimensional example:

self-dual

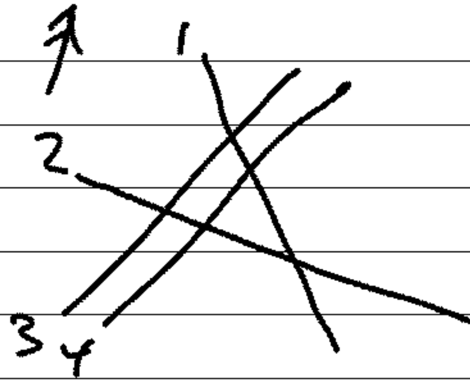




$\mathbb{C}^2/\mathbb{Z}_2 \qquad\qquad T^*\mathbb{P}^2$

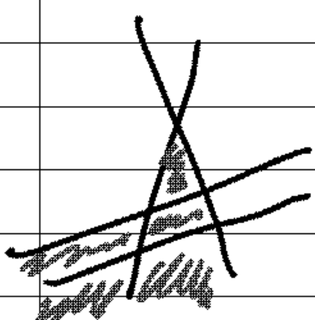


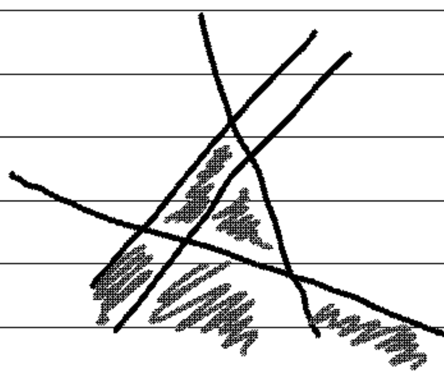
parallel
(ok for our genericity: can't have them coincide)



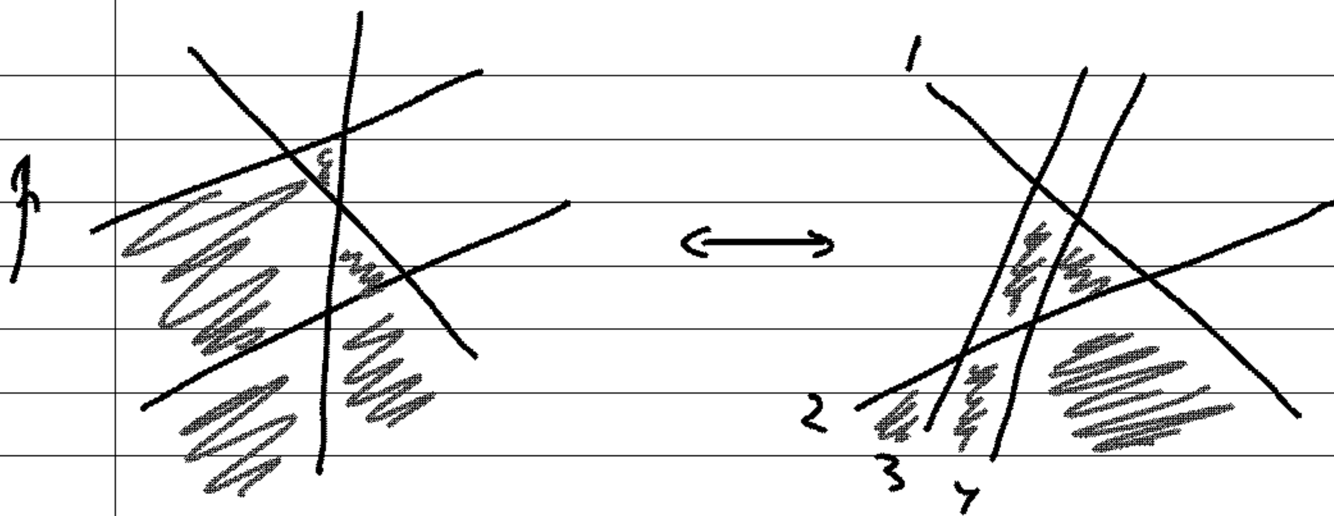
[These pictures will come from hyperplane arrangements <u>not</u> coming from quivers (quiver gives arrangement with hyperplanes $\leftrightarrow$ edges)]

Bounded chambers in this example





- different combinatorics of bounded chambers (though hyperplane combinatorics is the same) $\Rightarrow$ different algebras



Chambers $\longleftrightarrow$ toric varieties

$\cap$ Lagrangian

corresponding hypertoric variety

[LHS: changed χ but kept ξ fixed] [RHS: fix χ, change ξ: rotate]

Geometry

Imagine everything defined over $\mathbb{Z}$
(all our lines have rational slope)

$$V_{\mathbb{Z}} \to \mathbb{Z}^I \to \mathbb{Z}^I / V^{\mathbb{Z}}$$

Pontryagin/Cartier dual: $T' \leftarrow (\mathbb{C}^*)^I \leftarrow T$

(T might not be connected since $\mathbb{Z}^I/V_\mathbb{Z}$ might have torsion ---> will give rise to orbifold points...)

$$\begin{tikzcd} T' & (\mathbb{C}^*)^I \arrow[l] & T \arrow[l] \arrow[dl, "\chi \text{ character}"] \\ & \mathbb{C}^* \arrow[ul, "\xi", "\text{cocharacter}"'] & \end{tikzcd}$$

$T \curvearrowright \mathbb{C}^I$, take $T^*(\mathbb{C}^I/T)$:

ie have moment map $\mu : T^*\mathbb{C}^I \longrightarrow \mathfrak{t}^*$

$$\underbrace{\mu^{-1}(0) \cap T^*_x\mathbb{C}^I}_{\text{cotangent fiber}} = \underbrace{T_x(T\cdot x)^\perp}_{\text{conormal to } T\text{-orbit of } x}$$

$$T^*(\mathbb{C}^I/T) = \mu^{-1}(0)/T$$

Semistable locus: $U_\chi = \mu^{-1}(0)^{\chi\text{-ss}}/T \subset \mu^{-1}(0)/T$

$$= \{ x \in \mu^{-1}(0) : \exists \varphi \in Fun(T^*\mathbb{C}^I)_{\chi^\ell} \quad \ell > 0 \text{ s.t. } \varphi(x) \neq 0 \}$$

ie $U_\chi = \mathrm{Proj} \bigoplus_\ell \mathrm{Fun}(\mu^{-1}(0))_{\chi^\ell}$

<u>Theorem</u> (classical) For each chamber A $\exists$ lagrangian subvariety X_A s.t.

1. X_A is a toric variety with polytope A
2. $\bigoplus_{\substack{A,B \\ \text{bounded}}} H^*(X_A \cap X_B), *$ convolution algebra on pairs

$\left(\cong_{\text{morally}} HF^*(X_A, X_B)\right) \simeq A(V^\vee)$

algebra associated to $V^\vee$ ... center of algebra is H^*(hypertoric)

X_A always rationally smooth

$\nexists$ compact hypertoric varieties

<u>Koszulity</u>: Each chamber $\longleftrightarrow$ simple

Build Verma associated to cone coming down from any fixed point