Edward Witten: Boundary Conditions II

Note Title 3/6/2008

3d SCFT

Superconformal group $OSp(4|4)$

$$\underset{\underset{SU(2)\times SU(2)}{\overset{15}{}}}{SO(4)} \times \underbrace{Sp(4)}_{\text{conformal group of } \mathbb{R}^{1,2}}$$

Superconformal classical theories:
Fix a metric on $\mathbb{R}^{1,2}$ & ask for global SUSY $H \subset OSp(4|4)$

Solutions: get SUSY (but not conformal) solution for any hyperkähler manifold X:
study $\Phi : \mathbb{R}^{1,2} \longrightarrow X$
$\langle , \rangle$ metric on X

$$I = \int_{\mathbb{R}^{2,1}} \left(\langle d\Phi, * d\Phi \rangle + \ldots \right)$$

... would be conformal in 2d but not 3d:

not invariant under scaling of $\mathbb{R}^{2,1}$

$\tilde{\sigma}_t : (x,y,z) \longmapsto t\,(x,y,z) \quad t>0.$

$\mathbb{R}^{1,2} \longrightarrow \mathbb{R}^{1,2}$

Can compensate iff X is conical :

ie $\exists \; \sigma_t : X \longrightarrow X$ s.t.

$\sigma_t^* (\langle , \rangle) = t \langle , \rangle$

$\longrightarrow$ can compose $\tilde{\sigma}_t \circ \sigma_t$ & get scale invariance.

So for a conical hyperkähler manifold get scale invariance.

For superconformal invariance, need $SU(2)$ action on X rotating the complex structures.

e.g $X = \mathbb{R}^{4n} = \mathbb{H}^n$

left or right action of $SU(2)$ of units in $\mathbb{H}$ gives desired symmetry...

but in fact have $\underline{\underline{2}}$ such superconformal structures --- can identify either factor of our $SU(2) \times SU(2) \subset OSp(4|4)$ with this global $SU(2)$ & get a superconformal theory —
[one $SU(2)$ acts on X, one acts only on fermions]

"mirror symmetry" of Intriligator & Seiberg
— related by outer automorphism of $OSp(4|4)$.

<u>2d analog</u> SCFT of a Calabi-Yau has

$$\begin{array}{ccc} OSp(2|2) & \times & OSp(2|2) \\ \cup & & \cup \\ O(2) \times Sp(2) & & O(2) \times Sp(2) \\ \circlearrowleft & & \circlearrowleft \\ \mathbb{Z}/2 & \times & \mathbb{Z}/2 \end{array}$$

outer automorphism

A σ-model is a functor

$$\text{Calabi-Yau manifolds} \longrightarrow \text{SCFTs}$$
$$Y \longmapsto \sigma(Y)$$

Y, Z are <u>mirror</u> if
$$\sigma(Y) = \tau \circ \sigma(Z)$$
where $\tau = * \times 1$ $\quad * \in \mathbb{Z}/2$

(note $* \times *$ = just orientation reversal, a <u>classical</u> symmetry of CYs)

[Combo of outer aut of conformal group giving auto of theories & of geometric symmetry]

3d! want functor from geometries (eg quivers) to 3d SCFTs we'd get an interesting mirror symmetry story, combining the outer automorphism with the geometric construction.

Gaussian case: flat $\mathbb{R}^{4n}$

Can write $\mathbb{R}^{4n} = \mathbb{R}^{4p} \times \mathbb{R}^{4q}$, can apply automorphism to one factor or the other.

Other conical examples are <u>singular</u> $\Longrightarrow$ classical description fails for two reasons:

1 • σ-models in >2 dims not well behaved in quantum world

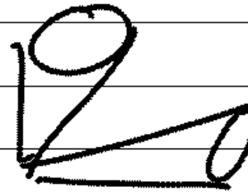
2 • have singularity! need to correct quantum theory.

$\rightsquigarrow$ need to define by more powerful methods.

Find eg X as a component of the moduli of vacua of the complete theory.

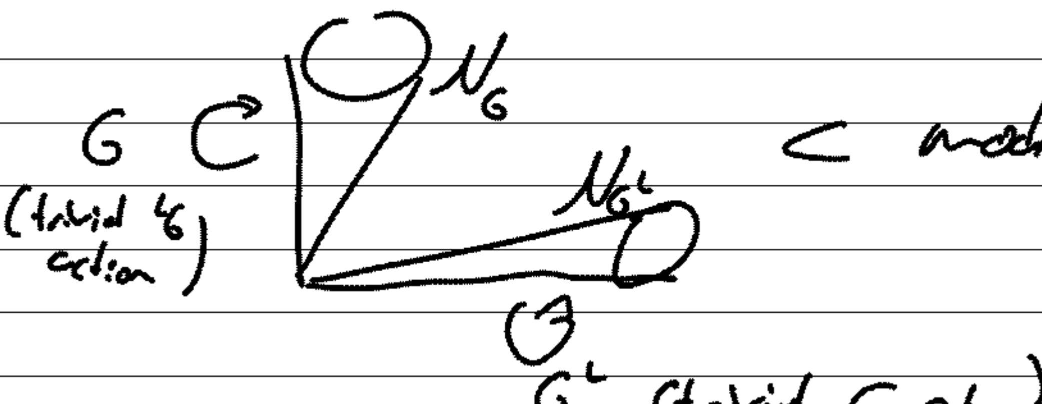
Often have several branches meeting

at the singularity:



Last time: SCFT $T(G, {}^LG)$

$G \times {}^LG$ symmetry, moduli of vacua with many irreducible components, in particular nilpotent cones $\mathcal{N}_G$, $\mathcal{N}_{G^L}$



$\subset$ moduli

Generic component:

$\mathcal{O}_\gamma :=$ nilpotent orbit of G

$\mathcal{O}_{\tilde{\gamma}} =$ " " of ${}^LG$

$\overline{\mathcal{O}_\gamma} \times \overline{\mathcal{O}_{\tilde{\gamma}}} \circlearrowleft G \times {}^LG$

generic component.

Each branch is a conical hyperkähler manifold with an $SU(2)$ inside the full $SU(2) \times SU(2)$ which rotates the complex structure, which were the $SU(2)$'s that appeared last time.

Self-mirror : $T(G, {}^L G) \simeq T({}^L G, G)$ under 3d mirror symmetry.

Classical theories with "global SUSY"

(subgroup preserving volume) $\qquad \cap$
$\qquad\qquad\qquad\qquad\qquad\qquad OSp(4|4)$

Any $G \rightsquigarrow$ minimal gauge theory, truncation of $N=4$ SYM:

Bosonic fields:

$$\underset{\text{gauge fields}}{A}, \quad \phi \in R \otimes \mathfrak{g}$$

(where $R = (3 \otimes 1)$ rep of $SU(2) \times SU(2)$)

(ie ϕ gives three scalar fields valued in $\mathfrak{g}$, acted on by $SU(2)$)

$$I = \frac{1}{g^2} \int \mathrm{Tr} \left(F \wedge * F + d_A \phi \wedge * d_A \phi + \sum_{i<j} [\phi_i, \phi_j]^2 + \text{fermions} \right)$$

Out of this we construct a hyperkähler manifold — the Coulomb branch — which has quantum corrections to its metric....

Add X a hyperkähler manifold with G action
— we'll take $X = \mathbb{R}^{4n}$, smooth,
— so that things will make sense!

Write $x : \mathbb{R}^{1,2} \longrightarrow X$ for our field

$\Rightarrow$ add to action $\langle d_A x, * d_A x \rangle + \langle \mu, \mu \rangle + |V(\varphi)|^2$

— forced by SUSY

Here $\langle \mu, \mu \rangle$ is the square of hyperkähler moment map

$\mu: X \to \mathfrak{g} \otimes \mathrm{Im}\,\mathbb{H} \simeq \mathfrak{g} \otimes \mathbb{R}^3$

[origin of hyperkähler moment map]

$V(\varphi)$: we have an action map

$\mathfrak{g} \longrightarrow \mathrm{Vect}(X)$,

$\varphi \longmapsto V(\varphi) \in \mathrm{Vect}(X) \otimes \mathbb{R}^3$

$\in \mathfrak{g} \otimes \mathbb{R}^3$

[One $SU(2)$ acts on φ
the other acts on X rotating complex structure, mirror symmetry will exchange them]

X here is really a linear representation

$\rho: G \longrightarrow Sp(2N) \hookrightarrow \mathbb{R}^{4N}$

— couple to gauge theory, ie x lives in $\mathbb{R}^{4N}$ bundle twisted by gauge group.

Problem Find "moduli space of vacua".

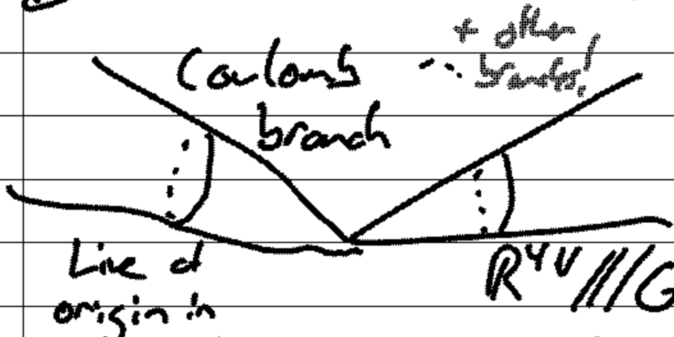
Classically: set energy to zero (as starting approx.) & divide by G.

$\Rightarrow$ set $\mu = 0$ & $V(\varphi) = 0$

$\Rightarrow$ so $x : \mathbb{R}^{1,2} \longrightarrow X$ must take value at a zero of $V(\varphi)$

& also $[\varphi_i, \varphi_j] = 0$.

Branches of solution space:



Higgs branch: $\varphi = 0$
$\Rightarrow$ hyperkähler quotient of X by G

conical hyperkähler manifold with $SU(2)$ rotating structures.

$\varphi \in \mathfrak{t} \otimes \mathrm{Im}\,\mathbb{H}$

up to gauge since $[\varphi_i, \varphi_j] = 0$

... dim looks like $3 \cdot \mathrm{rk}\, \mathfrak{g}$ ... will fix to make it hyperkähler —

get opposite $SU(2)'$ acting on φ &
on the Coulomb branch.

Naively Coulomb branch $\sim$ 3rk of
$\varphi \in \mathfrak{t} \otimes \mathrm{Im}\,\mathbb{H}$.

Suppose first φ is a <u>regular</u> triple in $\mathfrak{t}$:
ie the centralizer of φ in G is precisely
$T \Longleftrightarrow$ no Weyl group stabilizer

φ breaks structure group from G to T
.... at low energy gauge group
observed is T, other components become
massive

<u>Ex</u> $G = SU(2)$ $T = U(1)$
curvature F = 2-form.

$0 = dF$, Bianchi identity

$0 = d \star F$: Maxwell's equations

So locally $F = * \, d\sigma$ (3 dims!)
where $\sigma : \mathbb{R}^{1,2} \longrightarrow {}^{L}T$
& this relation holds on quantum level as well.

Classically the Coulomb branch is
$$\left.\begin{array}{l} \underbrace{\mathfrak{g} \otimes \mathbb{R}^3}_{\phi,} \times {}^{L}T \;/\; \text{Weyl} \\ \qquad [\phi_i, \phi_j] = 0 \end{array}\right\} \text{not conical!}$$
(flat hyperkähler manifold with $SU(2)$
action: $\mathbb{R}^3 \times \mathbb{R} \cong \mathbb{H}$
$SO(3)$: diagonal action of unit quaternions on $\mathbb{H}$

We started not with an SCFT but just SUSY field theory — lack of conformal invariance means it's ok Coulomb branch was not conical; Higgs branch was!

Other branches: reduction of fixed point sets of $V(\varphi)$ by stabilizers.

Moduli of quantum vacua deforms this picture: branches are the same but hyperkähler metric gets deformed — though not on Higgs branch, where nothing happens!

Coulomb branch is modified quantum mechanically. (Note $(\mathfrak{g} \otimes \mathbb{R}^3 \times {}^L T)/W$ already has various conical singularities!)
Classical metric on Coulomb branch
$ds^2 = \frac{1}{g^2} d\vec{\varphi}^2 + g^2 (d\sigma)^2 \qquad \sigma \in {}^L T$

Intrilligator – Seiberg: explicit example for G abelian

$G = U(1) \qquad X = \mathbb{H}^n$ with natural $U(1)$ action

[This is 3d analog of Seiberg-Witten theory for Coulomb branch — one loop correction + instantons]

$U(1)$ case: no instantons $\rightsquigarrow$ get extra $U(1)$ symmetry, $\sigma \longmapsto \sigma + \text{const}$, get <u>toric</u> hyperkähler manifold.

Explicit one loop correction:

$$ds^2 = \underbrace{\left(\frac{1}{g^2} + \frac{n}{|\vec{\phi}|}\right)}_{\tilde{g}^2} d\vec{\phi}^{\,2} + \tilde{g}^2 (d\sigma + n\omega)$$

special case of multi-Taub-NUT metric.

... <u>nontrivial</u> circle bundle over $\mathbb{R}^3$

$$\begin{array}{c} S^1 \hookrightarrow M \\ \downarrow \\ \mathbb{R}^3 \end{array}$$

ω = connection one form on the circle bundle;

The singularity at $\phi = 0$ looks like $\mathbb{R}^4/\mathbb{Z}_n$

Our SCFT comes by looking closely near singularity — ie replace the full space by $\mathbb{R}^4/\mathbb{Z}_n$.

- RG flow: our field theory is not invariant by homothety, can rescale by 1-parameter family & try to take limit to get CFT (doesn't always work!)

In this case an equivalent operation is taking $g \to \infty$: equivalent in 3d to taking homotheties.

$g \to \infty \Rightarrow \tilde{g}^2 = \frac{\phi}{n}$

$$ds^2 = \frac{n}{|\vec{\phi}|} d\varphi^2 + \frac{\phi}{n} (d\sigma + n\,\omega)$$

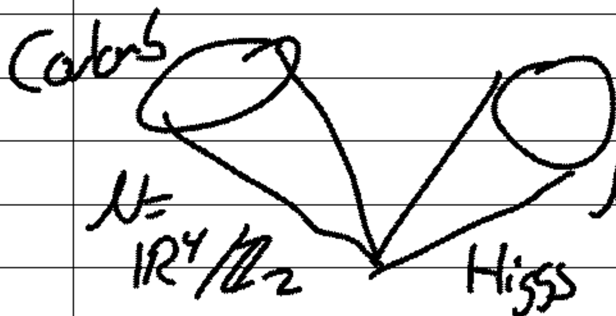
– conical hyperkähler metric, which is quantum version of the Coulomb branch.

e.g. $n=1$: $\mathbb{R}^4/\mathbb{Z}_n$ no singularity
– in this case Higgs branch disappears & Coulomb branch is smooth.

This story works for any G & sufficiently big symplectic representation.

Ex. $N=M=2$.

$\mathbb{H}^2 /\!/ U(1) = \mathbb{R}^4/\mathbb{Z}_2$ A_1 singularity
(quiver construction)
$= \mathcal{N}_{sl_2}$ nilpotent cone



$\mathcal{N} = \mathbb{R}^4/\mathbb{Z}_2 \circlearrowleft SU(2)$ visible
"$SU(2)$" rotates $\mathbb{C}$ structure

Two branches are both $\mathcal{N}$ but for different reasons — for Coulomb by a tricky quantum argument, for Higgs for evident reasons

Coulomb: $U(1) \times SU(2) \subset Sp(4)$ acts on $\mathbb{H}^2/U(1)$, get $SU(2)$ action for a dual $SU(2)$.

Maximal torus of this $SU(2)$ is dual to the gauge group torus $U(1)$.

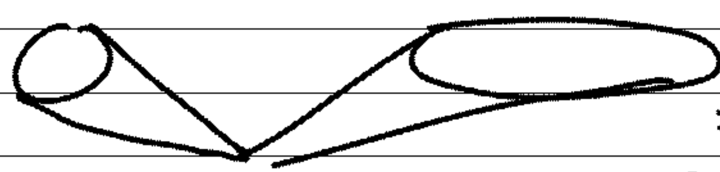
$$\Rightarrow \quad T(G, {}^L G) = T(SU(2), SU(2))$$

from $U(1)$ gauge theory

Higgs $= \mathcal{N}_G$ Coulomb $= \mathcal{N}_{{}^L G}$.

Same example for $N > 2$:

$\mathbb{R}^4/\mathbb{Z}_n$



$\mathbb{H}^n /\!/\!/ U(1)$

$=$ minimal nilpotent orbit of $\mathfrak{sl}(n)$

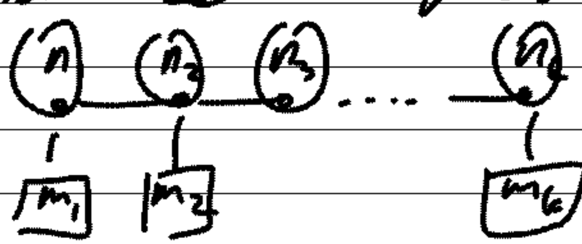
("sub"minimal really: not 0!)

$\mathbb{R}^4/\mathbb{Z}_n$ = Slodowy slice to the subregular nilpotent orbit of $SL(n)$.

<u>General Story</u> : Coulomb Slodowy slice to a nilpotent orbit of G & Higgs is closure of a nilpotent orbit for ${}^LG$.

G any compact group, R any suff. big representation in $Sp(2n)$ have a singularity leading to a SCFT

Special cases via quivers:



$N=2$ quiver (with bifundamentals)

framing:

"sufficiently big" (Nakajima)

let $w_i = m_i + (Cn)_i$ $\quad C$ = Cartan matrix

- want $w_i \geq 0$

ie want hyperkähler quotient sound
— set SCFT,
Mirror operation here merely exchanges m's & v's.

Quivers $\longrightarrow$ SCFT
$Q \longmapsto \sigma(Q)$

Q_1 & Q_2 are <u>mirror</u> if $\sigma(Q) = \tau \circ \sigma(Q')$

Seiberg-Intriligator: $\overset{n=1}{\bullet} - \underset{m=2}{}$ $w=2$ self mirror

- $G' = SU(N) = G''$ have theory $T(G', G'')$ with symmetry $G' \times G''$ & $SU(2)' \times SU(2)''$

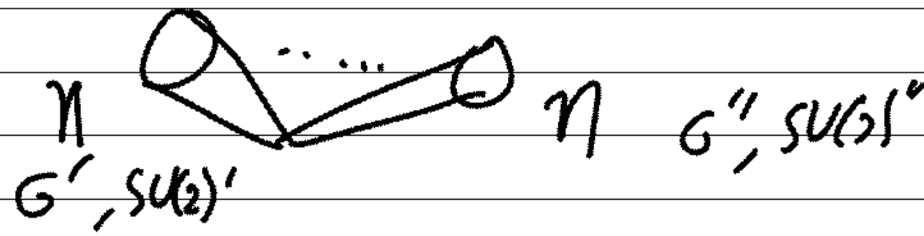
$(1) - (2) - \cdots - (N-1) - \boxed{N}$

only one $\square \Rightarrow U(N)$ symmetry

$\dim_{\mathbb{H}} \text{Coulomb} = rk\,(U(1) \times U(2) \times \ldots \times U(N-1))$
$= \frac{1}{2} N(N-1)$

$\dim_{\mathbb{H}} R - \dim G$ is also $\frac{1}{2}N(N-1)$

This quiver is self-mirror



Old: Higgs branch is $\mathcal{N}$

New: Coulomb branch is also $\mathcal{N}$

In general get mirror symmetry of nilpotent orbits & Slodowy slices of dual orbit (assigned to dual quiver)

Principal SL(2) in geometric Langlands

Why is geometric Langlands $\mathbb{Z}$-graded?

$N=4$ in 4dim!

$$PSU(4|4) > \underset{\substack{\wr \\ SU(4)}}{SO(6)} \times \underset{\text{conformal group}}{SO(4,2)}$$

Topological twisting breaks $SO(6)$ into a subgroup commuting with $SU(4)$

$$SO(6) \supset SO(2) \times SO(4)$$

Start with a 4d superconformal multiplet for $PSU(4|4)$: choose a priori any rank 6 real vector bundle V.

We'll take $V = TM_4 \oplus \mathbb{R}^2$.

$\mathbb{Z}$-grading comes from automorphisms

$SO(2)$, acts on everything,

$\mathbb{Q}$ hence theory is $\mathbb{Z}$ graded.

$V = \mathbb{R}^2 \oplus TM_4$

This $SO(2)$ sits in original $SO(6)$

& $SO(6)$ acts on the six scalar fields

$\vec{\phi} = (\phi_1, \dots \phi_6) \in \mathfrak{g} \otimes \mathbb{R}^6$

So $SO(2) \circlearrowright \sigma = \frac{\phi_5 + i\phi_6}{\sqrt{2}} \in \mathfrak{g}_{\mathbb{C}}$

Last time: three of the ϕ's had poles, labelled by homomorphisms $\mathfrak{su}_2 \to \mathfrak{g}$

Principal $\mathfrak{su}_2$ was dual to the elementary (Neumann) boundary conditions.

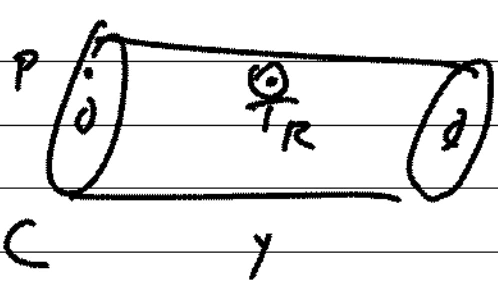
Two cases: σ does or does not have a pole $\rightsquigarrow$
2 appearances of Sl_2 in geometric Langlands-

If R = rep of ${}^LG$
$\longleftrightarrow$ orbit in affine Grassmannian on G, $\mathcal{O}_R$

$IH^*(\mathcal{O}_R)$ intersection cohomology is a $\mathbb{Z}$-graded vector space with principal Sl_2 action.

Where does $\mathcal{O}_R$ arise?

C curve $\ni p$ point



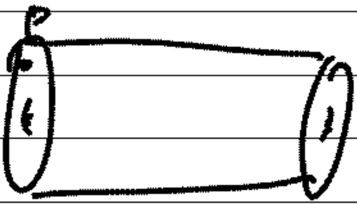
look at all Hecke modifications of type R for G

- T_R 't Hooft operators

Boundary condition: fix incoming bundle but allow any outgoing bundle

$\Rightarrow$ orbit $\mathcal{O}_R$ as space of modifications & space of physical states on the 3-manifold is the cohomology $IH(\mathcal{O}_R)$.

fix E

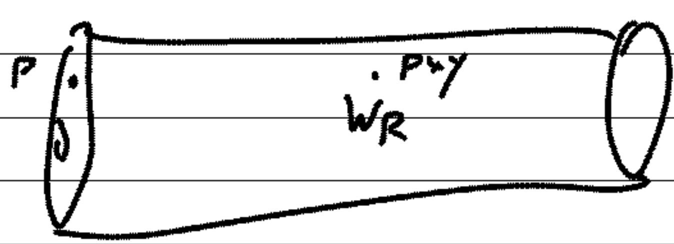


$\leftarrow$ allow any G-bundle here, i.e. Neumann boundary conditions on RHS

On other side we want to fix E: locally the same as Dirichlet conditions. (convenient to have ired stable bundle...) $\rightsquigarrow$ modified Dirichlet.

To understand this let's consider the dual picture

$C \times I$



W_R = Wilson operator of ${}^LG$

Dual boundary conditions:

Dual of Neumann $\rightsquigarrow$ σ has a pole

if boundary is $y = y_0$

$\rightsquigarrow$ $\sigma \approx \frac{1}{y_0 - y} \cdot t$

$t \in \mathfrak{g}_{\mathbb{C}}$ principal nilpotent

(dual to Neumann in B model $\rightsquigarrow$ σ won't have pole..)

[Other side have form of dual to Dirichlet]

<u>Physical Hilbert space</u>: principal Sl_2 rep is irreducible, so get frozen modes.

As a vector space just need to quantize W_R $\rightsquigarrow$ Hilbert space $= R$.

Really $SO(2)$ that acted on σ is lifted to $Spin(2)$ on fermions

— wrt $U(1) = Spin(2)$ σ has spin 2 ... $\longleftrightarrow$ even dimensionality of cohomology.

$\sigma \longmapsto e^{2i\alpha}\sigma$ $\quad e^{i\alpha} \in U(1)$

Can't change boundary condition = pole of σ — must modify by gauge transformation wrt which pole of σ has weight 2

$\Rightarrow$ $\mathbb{Z}$ grading of R by Cartan of principal Sl_2, normalized so σ has degree 2.

What about raising operator = principal nilpotent $\longleftrightarrow$ 2dim cohomology class of the affine Grassmannian? (acts on R as multiplication by $\sigma \sim \frac{1}{y_0 - y} + \ldots$)
principal nilpotent

$(\phi_1, \sigma) \sim \frac{h}{y_0-y}, \frac{e}{y_0-y}$
another scalar field

In fact full $H^*(Gr) \longrightarrow \mathrm{End}\, R$

– hints from Witten's youth, in Donaldson theory: (1988)

$N=2$ gauge theory had field σ,

P = invariant polynomial on $\mathfrak{g}$,

$P(\sigma)$ quantum field operator, commutes with Donaldson theory (& GL) differentials

$\sigma \longleftrightarrow$ curvature 2 form on universal bundle in Donaldson theory

M_4 $\quad P(\sigma(x))$ gives $2k$-dim cohomology class.

Solve by Seiberg-Witten theory!

get $P(\sigma_{eff}(x))$ in SW effective theory

.. actually don't want to act at a point

but look at $M_4 \times \mathrm{Bun}_G$

get $2k$ dim class on here,

Can integrate over cycles

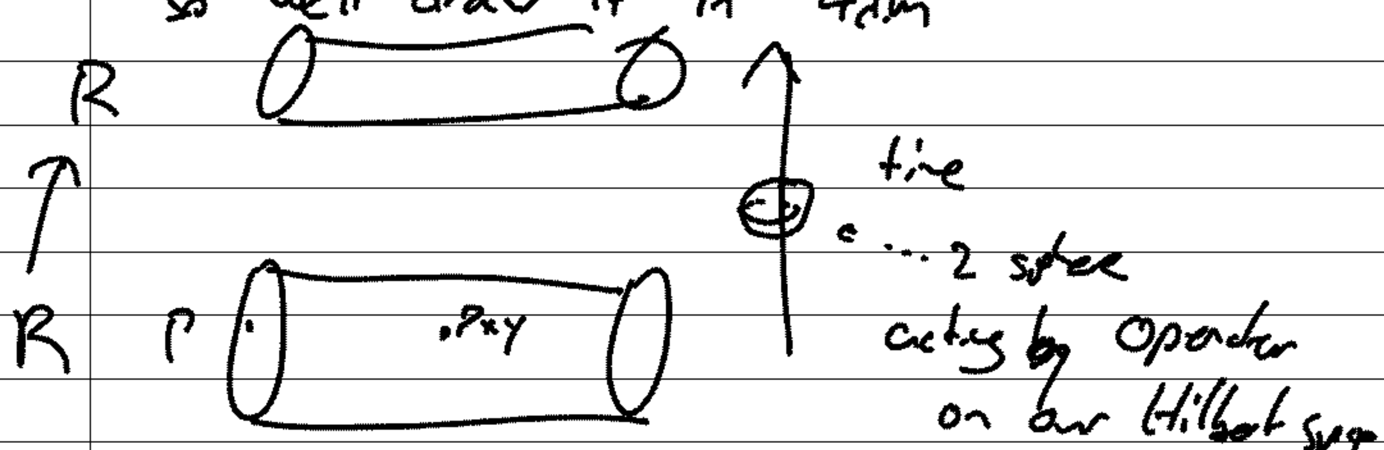
$\rightsquigarrow$ k form valued fields on spacetime.

Our case! a 2 form valued field

$$\left(\frac{\partial P}{\partial \sigma} F + \frac{\partial^2 F}{\partial \sigma^2} \psi\psi \right) = \mathcal{O}^{(2)}(P, \sigma)$$

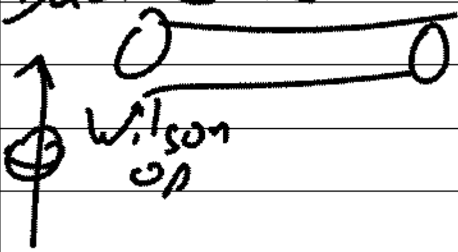
Action of $H^*(G)$ is not a functorial action but rather an action on the cohomology — ie localized in time!

So we'll draw it in 4dim



Operator : $\int_{S^2} \mathcal{O}^2(P, \sigma)$

Dualize the full picture:



Operator is $\int_{S^2} \mathrm{Tr}(\phi F + \psi\psi)$

Same expression on dual side

- fermions don't do anything, F vanishes due to electric charge

$\Longrightarrow$ operator is $\dfrac{\partial P^L}{\partial \sigma}$ $\quad$ $P(\sigma)$ corresponding invariant poly

- polynomial in principal nilpotent.

We have a map invariant polys $\longrightarrow H^*(\mathrm{Bun}_G)$

$\searrow H^*(\mathrm{Gr})$