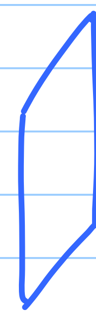
Witten "Boundary Conditions in $\mathcal{N}=4$ SYM"

Note Title 2/28/2008

$\mathcal{N}=4$ super Yang-Mills $PSU(4|4)$

on half space $\mathbb{R}^{1,3}_{+}$



$\mathbb{R}^{1,2}$

$PSU(4|2,2) \supset SU(2,2)$

conf group of $\mathbb{R}^{1,2}$

symmetry $OSp(n|4) \supset Sp(4)$

$\cap$ $\quad\quad\quad\quad\downarrow$

$PSU(4|2,2)$ $\quad SO(2,3) =$ conf. gp of $\mathbb{R}^{1,2}$

$n \leq 4$

max. symmetry is $n=4$

$OSp(4|4) \subset PSU(4|4)$

$SO(4) \times Sp(4) \subset SU(4) \times SU(4)$

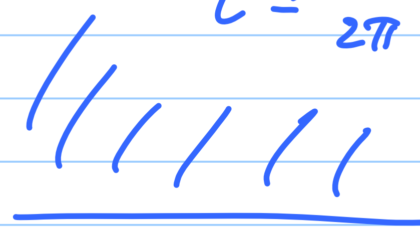
Outer Aut $(PSU(4|4)) = U(1)$

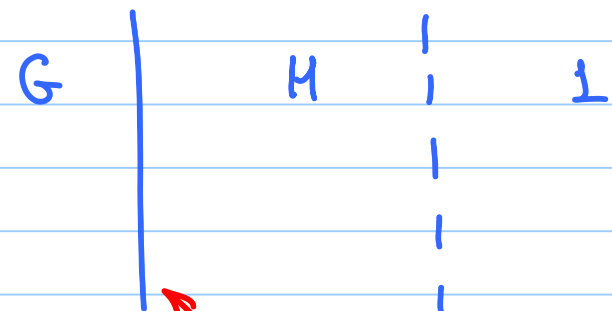
$\Rightarrow$ S^1 bdle

$\downarrow$

$\mathcal{H}$ = upper half plane

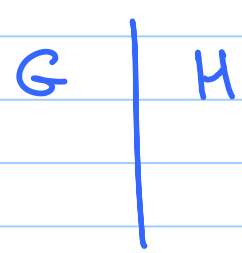
$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$$



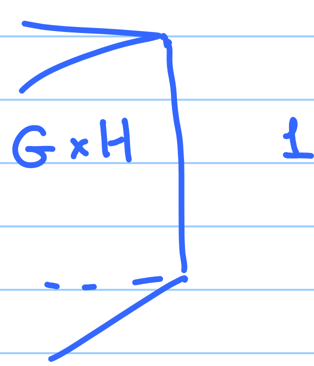


supersymmetric domain wall

dual to embedding $i: H \to G$



$\simeq$



* Some Simple SUSY Boundary Conditions:

1) Dirichlet: $F = dA + A\wedge A$ = curvature

$$F\big|_{\partial M_4} = 0 \leftarrow\text{-- } G\text{-bdle } E \to M_4$$

given a covariant trivialization on ∂M_4

2) Neumann: $*F\big|_{\partial M_4} = 0$

* Is Dirichlet for $G, \tau \Leftrightarrow$ Neumann for ${}^LG$, ${}^L\tau = -\frac{n_g}{\tau}$?

- Yes for $G = U(1)$
- No for non-Abelian G

$\Rightarrow$ Need to generalize both sides to get a duality-invariant picture

* first, generalize 2)

over bdry, have a G-bdle $E \to \partial M_4$

Pick any 3-dim'l superconformal QFT

w/ $OSp(4|4) + G$ symmetry

and make a "twisted" product

$\rightsquigarrow$ combined theory : $\mathcal{N}=4$ SYM coupled to 3d SCFT

* $\mathcal{N}=4$ super Yang-Mills

$$I = i\int \mathrm{Tr}\left(F\wedge * F + \sum_{i=1}^{6} D\varphi_i \wedge * D\varphi_i + \sum_{ij} [\varphi_i, \varphi_j]^2\right)$$

+ fermions

$$SU(4) \subset PSU(4|4)$$
$$\cong$$
$$SO(6)$$

classical vacua (preserve the metric)

$$A=0 \qquad \varphi_i \in \mathfrak{t} = \mathrm{Lie}(T)$$

$$d\varphi_i = 0 \quad , \quad [\varphi_i, \varphi_j] = 0$$

$$\Rightarrow \quad \mathcal{M} = \left(\mathfrak{t}^{\otimes 6}\right)/\mathrm{Weyl}$$

* on half space:

$$SO(6) \supset SO(4) \qquad PSU(2|2) \supset OSp(4|4)$$

SO(4) has two 3-dim reps (self-dual & anti-self-dual forms on $\mathbb{R}^4$)

V = 4-dimn'l SO(4) module

$$\Lambda^2 V = (\Lambda^2 V)^+ \oplus (\Lambda^2 V)^-$$

$\varphi = X \oplus Y$ $\qquad \Lambda^2 3 \simeq 3 \qquad \Lambda^2 \mathfrak{g} \to \mathfrak{g}$

$\phantom{\varphi = {}} 3 \quad\; 3' \qquad\qquad \varphi \in [3] \otimes \mathfrak{g}$

two different
real reps. of $SO(4)$

$X|$ not req'd to be zero

$$d\vec{X} + \vec{X} \times \vec{X} = 0 \quad \text{modified Neumann}$$

$$Y|_{\partial M} = 0 \quad \text{Dirichlet}$$

* Nahm's eqn.

$$\frac{d\vec{X}}{dy} + \vec{X} \times \vec{X} = 0$$

$$[\vec{X}, \vec{Y}] = [\vec{Y}, \vec{Y}] = d\vec{Y} = 0$$

* conformally inv't solution:

$$\vec{X} = \frac{\vec{t}}{y} \qquad \rho: su(2) \to \mathfrak{g}$$
$$\vec{t} = \rho(\vec{\sigma})$$

$$[t_1, t_2] = t_3 \ , \ \text{etc.}$$

generalizes Dirichlet

and classified by $\rho: su(2) \to \mathfrak{g}$

* consider the problem defined by $\rho: su(2) \to \mathfrak{g}$

require $\vec{X} = \frac{\vec{t}}{y}$ at $y \to 0$

also pick $\lim_{y\to\infty} \vec{X}(y) = \vec{X}_\infty$ "vacuum at ∞"

$\mathcal{M}(\rho, \vec{x})$ hyper-Kähler as a complex manifold,

$$\mathcal{M}(\rho, \vec{X}) = \mathcal{S}_{\rho, x} \quad \text{Slodowy slice}$$

where $x = X_1 + i X_2 \in \mathfrak{g}_{\mathbb{C}}$ and

$\rho \leftrightarrow$ nilpotent orbit $\mathcal{O}_\rho$ = orbit of $t_+ = t_1 + i t_2$

Summarizing :

modified here

fix vacuum $\vec{X} \to \vec{X}_\infty$ at ∞

$\mathcal{M}$ = moduli space of SUSY vacua

hyper-kähler

* Examples of Slodowy slices
(x = principal nilpotent)

i) $\rho = 0 \qquad \mathcal{S}_{(\rho, x)} = \mathcal{O}_x = \mathcal{N}$ = nilpotent cone

naive Dirichlet

⋮

ii) ρ = principal nilpotent

$\mathcal{S}_{(\rho,x)}$ = point $\Rightarrow$ dual to naive Neumann

iii) $T(G, {}^LG)$ action of $(G \times {}^LG)_{ad}$

moduli space of vacua

General branch : $\overline{\mathcal{O}_\rho} \times \overline{\mathcal{O}_{{}^L\rho}}$

$\rho : su(2) \to \mathfrak{g}$

${}^L\rho : su(2) \to {}^L\mathfrak{g}$

$\rho \longleftrightarrow {}^L\rho$ order reversing

e.g. subreg $\longleftrightarrow$ minimal