E. Witten - Duality from Six Dimensions 2

Note Title 2/14/2008

<u>Remark</u>. In n dimensions, superconformal group G has even part $Spin(2,n) \times H$.

Which of these can we get by reducing dimensions?

Pick τ : $Spin(2n) \circlearrowleft$ involution, fixing $Spin(2,m)$ $m<n$

<u>If</u> we can lift τ to $G \circlearrowleft$ $\Rightarrow$ G^τ = superconformal group in dim m.

<u>However</u>: the 5-dim group <u>can't</u> be obtained from the 6-dim group this way.

Last time: for each ADE gp $\rightsquigarrow$ 6d superconformal theory, with group $OSp(2,6|4)$ $[SO(2,6) \times Sp(4)]$

Formulation on 6-manifold M w/ a little extra data:

let $Z(G)$ = center of group G of ADE type (simply connected).

Have a form $Z(G) \times Z(G) \to U(1)$ self dual pairing

$\Rightarrow$ pairing $H^3(M_6, Z(G)) \times H^3(M_6, Z(G)) \to U(1)$
using intersection pairing
$\rightsquigarrow$ make a Heisenberg group

Pick a vector in the rep of the Heisenberg group $\Longleftrightarrow$ a maximal split subgroup under above pairing .... this is the extra structure we need on M.

[2d analog: level 1 current algebra on Σ = 2-manifold, have pairing
$H^1(\Sigma, Z(G)) \times H^1(\Sigma, Z(G)) \to U(1)$
& need to polarize to define spaces of conformal blocks.]

If $M_6 = M_5 \times S^1 \implies$ get a low energy / long distance description of our theory by gauge theory on M_5, & conformal invariance $\implies$ coefficient of action $I = \frac{1}{r}\int \mathrm{Tr}\, F \wedge * F$

is <u>inverse</u> radius ... opposite to dim reduction of a gauge theory!

Refinement: want to consider states of 6d theory not invariant under S^1 rotation, ie states of different momenta — hard without classical description ...

A) Let $M_6 = \underset{\text{time}}{\mathbb{R}_+} \times M_4 \times S^1_r$

to get hamiltonian interpretation (Hilbert space)

Two operators are "energy" $H \longleftrightarrow -i\frac{\partial}{\partial t}$

momentum $P \longleftrightarrow -i\frac{\partial}{\partial \theta}$

$\theta \in [0, 2\pi r]$ parameter on S^1

What does P correspond to in gauge theory description?

$\mathcal{H}$ = Hilbert space (quantizing all the fields, connection A etc. on M_4)

[SUSY $\Rightarrow$ $H - P = \sum Q_i^2$, $H + P = \sum \tilde{Q}_i^2$

so H is a sum of squares:
superconformal group on M_4 has odd generators, either by twisting or if M_4 is hyperkähler, & their commutators give combos of the even generators, which are only H, P generically as geometric symmetry & $Sp(4)$ R-symmetry: acts on spinor part]

... Hilbert space comes from quantizing fields on M_4, which breaks up as a direct sum

$$\mathcal{H} = \bigoplus_n \mathcal{H}_n$$

$\mathcal{H}_n$ = quantization of connections on bundles with $c_2 = n$.

P acts by multiplication by $\frac{1}{2\pi} \cdot n$ $\left(\begin{matrix}\text{up to some}\\ \text{fixed constant}\end{matrix}\right)$

- momentum along circle matches up with c_2 of bundles.

SUSY $\Rightarrow$ $H \geq |P|$ for all states

& $(H-P)|\psi\rangle = 0$

$\Rightarrow$ some SUSY $Q_i|\psi\rangle = 0$

$(H+P)|\psi\rangle = 0 \Rightarrow$ some $\bar{Q}_i|\psi\rangle = 0$

So these states are very special! BPS states

If M_4 is hyperkähler, then

$$\left\{\psi \;\middle|\; P\psi = \frac{1}{2\pi R} n\psi \;\&\; (H-P)\psi = 0\right\}$$

is the cohomology of the instanton moduli space. (we're suppressing center of group / w_2 invariants)

M_4 hyperkähler $\Rightarrow$ moduli of instantons is hyperkähler $\Rightarrow$ cohomology of moduli space carries an $Sp(4)$ action ("Lefschetz")

Rather the local superconformal symmetry group is $OSp(2,6/4) \supset Sp(4)$.
On a hyperkähler manifold this survives

By "cohomology" we mean space of L^2 harmonic forms: it arises from relation
$H - P = \sum Q_i^2$: any state annihilated by $H-P$ must be annihilated by all the Q_i.

On hyperkähler manifold a harmonic form is annihilated by 8 different operators, transforming in fundamental rep of $Sp(4)$.

Problems with instanton moduli: for

M_4 noncompact $\longrightarrow$ instanton moduli noncompact : big instantons.

Also have problems with small instantons: moduli is very singular. Can make sense for types A, D: type A use noncommutative deformation of $U(n)$ instantons (Nekrasov etc)

On ALE space can use ADHM construction formally to define our space of harmonic forms by symplectic reduction: quantum reduction is our <u>definition</u> of the cohomology.

B. Instead of $M_6 = M_5 \times S^1$ take a fibration $S^1 \longrightarrow M_6$

$$\begin{array}{ccc} S^1 & \longrightarrow & M_6 \\ & & \downarrow \\ & & M_5 \end{array}$$

Describe the conformal structure via a Riemannian metric on M_5 & on S^1 (ie a radius r) & a $U(1)$ connection on the circle bundle.
[assume orientable for today]

Still get a 5-dimensional gauge theory on M_5 but with extra term in the action

let f = curvature 2-form of the S^1 connection

& let $CS(A) = \frac{1}{4\pi} Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$
= Chern-Simons 3-form of A
- defined up to 3-form with integer periods
(for trivial bundle)

$$I = \frac{1}{r} \int_{M_5} Tr(F \wedge * F \ldots) + \int_{M_5} f \wedge CS(A)$$

$\Rightarrow$ $\exp iI$ is well defined
(due to integrality of f)

In general the path integral has a factor
$\exp\left(-i \int_{M_5} f \wedge CS(A')\right)$:

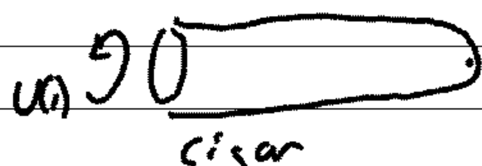
... have a $U(1) \times G$ gauge theory on M_5
where $U(1)$ acts on extra circle
& have an invariant form in $Lie(U(1) \times G)$
$\Rightarrow$ get a well defined Chern-Simons 5-form

if $c_1 \neq 0$ invariant depends not just on
curvature but on holonomy of the circle bundle.

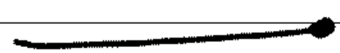
Now allow S^1 to degenerate at some points!

consider $\begin{array}{c} M_F \\ \downarrow \pi \\ M_S \end{array}$ generically an S^1 fibration
---- eg if M_F has a generically free $U(1)$ action,
$M_S = M_F / U(1)$

Two cases of interest:

1. 

$\times M^4$


$\times M^4$ quotient: $M^4 \times$ half-line

= SUSY YM on M^4 with a BPS boundary condition (don't know which one

2. S^1 action on $\mathbb{R}^4 = \mathbb{C}^2$ by complex scalars
- isolated fixed point at origin
(this $U(1) \subset U(2)$ which commutes with an $SU(2)$, rotating three complex structure):

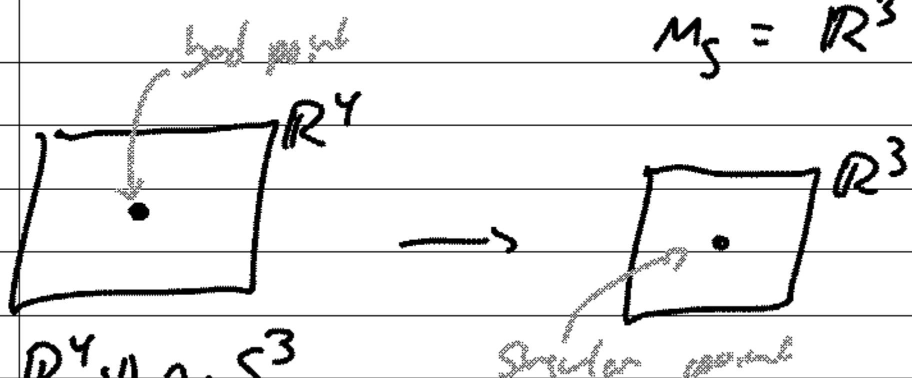
Our $U(1)$ is left multiplication by i on $\mathbb{H}$, commutes with right i, j, k multiplication.

Quotient by $U(1)$ is $\mathbb{R}^3$, & hyperkähler moment map is

$$\mathbb{R}^4/U(1) \xrightarrow{\sim} \mathbb{R}^3$$

[$\vec{x}$ = coords on $\mathbb{R}^3$ = $\bar{z}\vec{\sigma}z$ in terms of $SU(2)$ generators]

[We'll consider
$$\begin{array}{c} M_\delta = \mathbb{H} \times \Sigma \\ \downarrow \\ M_S = \mathbb{R}^3 \times \Sigma \end{array}$$
]



$$\mathbb{R}^4 \setminus 0 \sim S^3 \xrightarrow[\text{w/ } c_1 = 1]{\text{Hopf fibration}} S^2 \sim \mathbb{R}^3 \setminus 0$$

(up to homotopy)

In standard hyperkähler metric on $\mathbb{R}^4$, the radius of $S^1 \to \infty$ at ∞

But have another hyperkähler metric where the radius is fixed at ∞:

TN = Taub-NUT : let $V = 1 + \frac{1}{|x|}$

$$ds^2 = \frac{1}{V}(\underbrace{d\psi + \vec{\omega}\cdot d\vec{x}}_{\alpha})^2 + V d\vec{x}^2$$

$\vec{x}$ = moment map to $\mathbb{R}^3$

α = connection one form on the $U(1)$ bundle: restriction to each fiber is $d\psi$ & $d\alpha = \pi^* f$.

radius^2 of circle is $\frac{1}{V}$ so has fixed value at ∞

Related L^2 harmonic 2-form on Taub-NUT (unlike on $\mathbb{R}^4$ w/ standard metric) which is $\frac{\pi^* f}{V} = \frac{d\alpha}{V}$

$$\left[f = * d \frac{1}{|x|} \right]$$

L^2 cohomology is one dim! normalize $\pi^* f$ to have integer periods of ω to get integral cohomology:

$H^2_{L^2 \text{ harmonic}}(TN, \mathbb{Z})$:

generator $(x, x) = 1$ — looks like $U(1)$ weight lattice

The metric is smooth at 0

Look in two ways at $M_6 = \mathbb{R} \times S^1 \times TN$:

A) : $M_6 \longrightarrow \mathbb{R} \times TN$, forget S^1 factor

$\Rightarrow$ gauge theory on $\mathbb{R} \times TN$

BPS states $= H^*_{L^2\text{-harm}}(\mathcal{M}_{inst}(TN, G))$

B) $M_6 \longrightarrow (\mathbb{R} \times S^1) \times \mathbb{R}^3$ via moment map $TN \longrightarrow \mathbb{R}^3$

$\Rightarrow$ gauge theory on $\mathbb{R} \times S^1 \times \mathbb{R}^3$ with some correction on $\mathbb{R} \times S^1 \times \{0\}$ from singularity.

M_f
$\downarrow$
$M_5 \supset \Sigma$ = 2-manifold of "bad points"
$= M_f^{U(1)}$:

set gauge theory on M_5 corrected along Σ.

Factor $\exp\left(-i\int f \wedge CS(A)\right)$ is not gauge invariant in our singular situation! $df = \delta_\Sigma$ not closed on Σ!

[we're assuming bundle is topologically trivial]

Under gauge transformation ε

$$CS(A) \longrightarrow CS(A) + d(\mathrm{Tr}\, \varepsilon\, dA)$$

$\therefore$ $\exp\left(-i\int f \wedge CS(A)\right)$ not well defined as a <u>number</u> but is defined as a section of a line bundle $\mathcal{L}^{-1} \longrightarrow$ space of connections ($\mathcal{L}$ = det bundle) on Σ

So along Σ "there are degrees of freedom that cancel the anomaly" –

ie there is a quantum theory w/ G symmetry whose partition function is $\mathcal{L}$-valued (anomaly!)
(ie can twist by any bundle with connection)
... couple to the connection A

[$U(n)$ case: free fermions have this same kind of anomaly.]

The current algebra of level 1 is such a system
..... level 1 WZW model.

<u>Hamiltonian version</u>

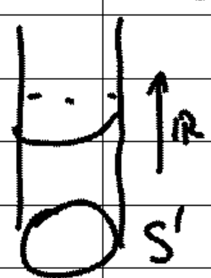
$$\begin{array}{c} \mathbb{R} = \text{time} \\ S^1 \times \mathbb{R}^3 = \text{Space} \\ \cup \\ S^1 \times pt \end{array}$$

$\Rightarrow$ see representation theory of loop group at level 1:

– (Gauge theory on $\mathbb{R}^3 \times S^1$) $\times$ (rep of LG at level 1)

& space of BPS states comes from the second factor : ' cohomology of instanton moduli space = level 1 rep of LG!

Hamiltonian picture! On $S^1 \times \mathbb{R}^3$ have gauge fields everywhere & rep of LG at $S^1 \times 0$.



This gives the Hilbert space of a two dim theory which makes sense on any surface Σ : it better, since any Σ can arise from our construction. ($\Rightarrow$ 2d CFT).

Space of lowest energy / BPS states doesn't feel the $\mathbb{R}^3$, just get the rep of the loop group: see a chiral CFT on the BPS states:

"$\frac{\partial}{\partial \bar{z}}$"? $\leftarrow$ $(H-P)(\text{Rep of } LG)=0$

[the way we broke SUSY means we get
only a chiral half ......]

On Taub-NUT: TN not simply connected at ∞:
should look at instantons with fixed holonomy

to get a good problem, & problem is best
behaved if holonomy is regular
(centralizer is a torus).

$\mathbb{R}^3 \times S^1$ picture: can fix holonomy around S^1
$\rightsquigarrow$ get representation of a
twisted loop group, with an inner twist (ie
still $\cong$ loop group).

$g(\theta+1) = h\, g(\theta)\, h^{-1}$.

We've broken G symmetry on the representation R
to the maximal torus (preserves the twisted
loop group): centralizer of h acts
& we're assuming $Z_G(h) = T$

Now decompose rep. R into T weights:

$R = \bigoplus_\alpha R_\alpha$ .... but α only lies in a torsor: $\alpha \in$ affine version of the weight lattice, i.e. we move h around a loop undergo shift in labelling.

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<u>Generalization</u> $\mathbb{R}^4 \rightsquigarrow \widetilde{\mathbb{R}^4/\mathbb{Z}_n}$ hyperkähler resolution

$\wr \qquad\qquad \wr$

$TN \rightsquigarrow TN_n$ multi Taub-NUT

TN_n hyperkähler 4-manifold with $U(1)$ symmetry $(\mathbb{Z}_n \subset U(1))$

$\mathbb{R}^4/\mathbb{Z}_n \longrightarrow \mathbb{R}^3$ hyperkähler moment map

Metric still has to have the form

$$ds^2 = \frac{1}{V}(d\psi + \vec{\omega}\cdot d\vec{x})^2 + V d\vec{x}^2$$

for <u>some</u> V $\qquad [f = * dV]$

We'll take $V = 1 + \sum_{i=1}^{n} \frac{1}{|\vec{x} - \vec{x}_i|}$

get smooth 4-manifold, singularities of V all look locally like the $n=1$ example.

If we erase the "$1+$" in V we get the ALE space, where radius of the circle increases at ∞, while in TN_n radius is fixed at ∞

$H^2_{\text{ordinary}} = \Gamma_{\text{root}}(A_{n-1})$ not unimodular

$\mathbb{H}^*_{L^2} = H^2_{L^2\text{-harmonic}} = \Gamma_{wt}(U(n)) = \langle x_1, \dots x_n : (x_i, x_j) = \delta_{ij} \rangle$

unimodular but not even

Now run same story as before:

$M_r = \mathbb{R} \times S^1 \times TN_{n; x_1 \dots x_n}$ $\quad x_1, \dots x_n \in \mathbb{R}^3$

A • $M_6 \longrightarrow \mathbb{R} \times TN_{n;x_1,\dots x_n}$

BPS states $= H^*_{L^2}(\mathcal{M}_{\text{inst}}(TN_{n;x_1,\dots x_n}))$

B • $M_6 \longrightarrow \mathbb{R} \times S^1 \times \mathbb{R}^3$ via $TN_n \longrightarrow \mathbb{R}^3$

$\Rightarrow \mathbb{R} \times S^1 \times \mathbb{R}^3_{x_1,\dots x_k}$ ie $\mathbb{R}^3$ with k distinguished pts

On $\mathbb{R} \times S^1 \times \{x_i\}$ have a current algebra at level 1:

$$\mathcal{H} \text{ "=" } \left(\begin{matrix}\text{quantize}\\ \text{gauge fields}\end{matrix}\right) \otimes \bigotimes_{i=1}^{k} R_{(i)}$$

$R_{(i)}$ = reps of LG ... which ones?

remember we have a finite Heisenberg group coming from $H^3(M_6, Z(G)) \times H^3(M_6, Z(G))$

In our case:

$$\| \quad H^2_{L^2}(TN) \otimes \Gamma^{wt}(G)$$

So our polarization ends up telling us which reps of loop group to take

... our 6-dim space $L^{1/3}(\ )$
$$\begin{array}{c} \| \\ H^1(\Sigma, Z(G))^{\otimes n} \end{array}$$

So 6-d problem reduces to n copies of the $U(1)$ story.

[$L_0 = C_2$, also have relative first chern class giving rest of torus action]

When some x's coincide get higher level reps at those points:

$x_i = x_j \Rightarrow$ get level 2 module for G

$x_i \longrightarrow x_j$ decompose level 1 $\otimes$ level 1
$= \bigoplus_{\text{irreps}}$ level 2 $\otimes$ rep of coset algebra (fd. L_0 grading)

So tensor of reps $\Leftrightarrow$ collisions of x's