Edward Witten - Duality from Six Dimensions III

Note Title 2/21/2008

Last time: 6d CFT of type G (ADE)
on multi-Taub-NUT space
- complete 4dim hyperkähler manifold

$$ds^2 = \frac{1}{V}(d\varphi + \vec{\omega}\, d\vec{x})^2 + V d\vec{x}^2$$

$$V = 1 + \sum \frac{1}{|x - x_k|}$$

$TN_{x_1,\dots,x_k} \longrightarrow \mathbb{R}^3$ quotient by S^1 symmetry:

[recall

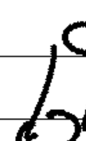
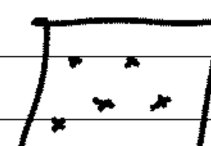
6 dim

 $\times TN_{x_1 \dots x_k} \Longrightarrow$

5 dim

A) $\uparrow \times TN_{x_1 \dots x_k}$

$H^*(M_{inst}(TN_{x_1 \dots x_k}))$

B)  $\times$  $\mathbb{R}^3$

bad points]

2 definitions of the 6d CFT:

- $\mathrm{II}B$ on $M_6 \times \mathbb{R}^4/\Gamma_G$ $\quad \Gamma_G \subset SU_2$

... we then took $M_6 = \text{cylinder} \times \underbrace{TN_{x_1 \ldots x_k}}_{\text{resolved } A_k \text{ singularity}}$

- so we're studying $\mathrm{II}B$

on cylinder $\times$ $\widetilde{A_k \text{ singularity}}$ $\times$ G singularity

but we didn't resolve the G singularity, only the A_k singularity $\rightsquigarrow TN$.

Today we'll resolve also the G singularity, & get an easier description.

$\mathbb{R}^4/\Gamma_G$ $\rightsquigarrow$ has "nontrivial" (ie non Gaussian) CFT -

If we resolve $\rightsquigarrow \widetilde{\mathbb{R}^4/\Gamma_G} \Rightarrow$ get theory which has a long distance description by "classical fields" — Gaussian theory.

Simplest QFTs : become Gaussian in UV: short distance behavior is Gaussian, so described by Lagrangian which has quadratic + perturbation.

$OSp(2,6|4) \supset Sp(4) \longrightarrow SO(5)$ symmetries of the fields.

Bose fields: • scalar field : $\phi \in \mathfrak{t} \otimes \mathbb{R}^5$

$\mathfrak{t} = \mathrm{Lie}(T \subset G)$

ϕ is regular if Weyl group acts freely, this is where we have a good description, nonregular case is the one we're interested in.

$\{\phi\}/W =$ vacua of the theory.

Write $\phi \in \mathfrak{t} \otimes (\mathbb{R}^3 \oplus \mathbb{R}^2)$

$\mathrm{Im}\,\mathbb{H}$: these were the components we saw before

$I = \int (d\varphi \wedge * d\varphi) + \dots$

• other bose field: connection B on a T-gerbe

B assigns to each surface $S \subset M_6$
a holonomy $\mathrm{Hol}_S(B) \subset T$

Curvature $H = dB$ obeys Bianchi $dH = 0$
Unusual feature: H is <u>self-dual!</u> $H = *H$

($*^2 = +1$ in signature $(1,5)$ so consistent
with H being real; this theory has
cousins in dimensions $4k+2$, nonlinear
versions known only in dim $2, 6$)

Self-duality follows from $OSp(2,6|4)$ symmetry
- If we didn't care about $H = *H$ would
have obvious action $I = \int H \wedge *H$,
get a reasonable theory...

<u>2d case</u> λ = 1-form, $d\lambda = 0$ $\lambda = *\lambda$
if we drop self-duality: locally $\lambda = d\varphi$
for scalar field φ, $I = \int \lambda \wedge * \lambda = \int d\varphi \wedge * d\varphi$
& φ is $U(1)$ valued.

Space of classical solutions (almost) decomposes into a sum of self-dual & anti-self dual parts.

If we do ask H (or λ) self dual
$\Rightarrow$ action $\int H \wedge * H = 0$! so can't
write classical action. But quantum theory
for all H's does decompose as a tensor product
(or sum thereof) of SD & ASD parts...

Write $M_r = \mathbb{R} \times M_S$, want to write
a Hilbert space associated to M_S by quantization.

Would like $M_S \rightsquigarrow \infty$-dim symplectic
manifold of possible B's.
Write δB = exterior derivative :
δB is a 2 form on M_S $\otimes$ 1-form on space of B's
B " " " " " $\otimes$ function " " "

Symplectic form $\quad \omega = \int_{M_S} \delta B \wedge d\, \delta B$

$$= - \int_{M_S} \delta B \wedge \delta H$$

This symplectic space <u>doesn't</u> come from the

extreme points of a Lagrangian!
Can quantize this, giving a Hilbert space

[cf arxiv 0712.0157, particularly ref. 3
by Henningson hep-th/1011150]

To define partition function:
2-d version, scalar field φ with $df = *d\varphi$
(really circle valued or torus valued $\longleftrightarrow$ lattice Λ)
partition function on T^2 is

$$Z = \frac{\Theta_\Lambda(q)}{\eta(q)^{\mathrm{rk}\,\Lambda}} \qquad \frac{\text{theta function}}{\text{eta function}} \text{ of lattice.}$$

(here we're taking Λ even lattice)

Applying modular transformations get θ functions
of cosets of Λ^*/Λ

Elementary (non self dual) theory has

$$Z^{\text{elem.}} = \sum_{\substack{\alpha \\ \text{cosets} \\ \Lambda^*/\Lambda}} \frac{|\theta_{\Lambda_\alpha}|^2}{|\eta^{\mathrm{rk}\,\Lambda}|^2} = \sum_\alpha \frac{\Theta_{\Lambda_\alpha}}{\eta^{\mathrm{rk}\,\Lambda}} \cdot \frac{\overline{\theta}_{\Lambda_\alpha}}{\overline{\eta}^{\mathrm{rk}\,\Lambda}}$$

For any genus we take the Heisenberg group for $H^1(\Sigma, \Lambda^*/\Lambda)$ & its irreducible module parametrizes the Θ_α's.

In 6 dimensions : Have gerbe for T connected abelian Lie group, with Λ = lattice of periods, & the theory factorizes into a direct sum of tensor products, & we're picking out a term in this factorization
($\Longleftrightarrow$ picking out level 1 representation in ADE case)
The choices of these summands will tell us on reducing to four dimensions which global form of the group G (adjoint, ...) we get.

Take $M_6 = M_5 \times S^1_R$ (circle of circum. $2\pi R$)

Self dual 3-form (S^1 invariant) has form $0 \leq \theta \leq 1$
$[F \in \Omega^2(M_5)]$

$$H = F \wedge d\theta + *_6(F \wedge d\theta)$$ (S^1 is spacelike)

$$= F \wedge d\theta + *_5 F$$

$$dH = 0 \Rightarrow dF = *dF = 0$$

& periods of F are in same lattice as those of H $\rightsquigarrow$ F is a connection on a T-bundle.

Can now take Lagrangian $\mathcal{L} = \not{H} \cdot \int F \wedge * F$

$$\int_{M^5 \times S^1} (F \wedge d\theta) \wedge * (F \wedge d\theta) = \frac{1}{R} \int_{M^5} F \wedge * F$$

.... curvature comes with a $\frac{1}{R}$ in 5 dim.

The gauge theory partition function now gives a nonholomorphic Θ function

$$\left(\frac{\Theta}{\det(\cdot\cdot)} \right)_{\text{gauge}} .$$

Also have a 6 dim theta function for the nonself dual theory & they agree [or rather a summand agrees] if we put the $\frac{1}{R}$ in front of the 5d action

2 dimensional theory

Consider a theory in 2d with many scalar fields φ_i, some obey $d\varphi_i = * d\varphi_i$ L
others $d\varphi_j = - * d\varphi_j$

More succinctly: Λ even integer lattice of signature (p,q)
(Narain) For each choice h: $\Lambda \underset{\mathbb{Z}}{\otimes} \mathbb{R} = \underset{\text{pos def}}{V_+} \oplus \underset{\text{neg. def}}{V_-}$

there's a 2-dimensional theory with partition function [in genus 1]

positive: $\dfrac{\Theta_\Lambda(q)}{\eta(q)^{rk}}$ indefinite: has a theta function associated to decomposition h of Siegel:

$$\frac{\Theta_{\Lambda,h}(q,\bar{q})}{\eta(q)^p\,\overline{\eta(q)}^q}$$

[genus g: theta function for Jacobian $\otimes \Lambda$]

If $\Lambda = \Lambda_+ \oplus \Lambda_-$ (ie h respects a decomposition over $\mathbb{Z}$)
$\Rightarrow$ this Θ is a product of hol. & antihol. factors

If $p=q$ can derive this Siegel function from a Lagrangian

Now consider $M_6 = \Sigma \times M_4$
Σ Riemann surface.

The theory of the T-valued self-dual gerbe in 6 dimensions gives for Σ "big" an effective 2d CFT on Σ for each M_4 [we'll assume $H^{odd}(M_4)=0$ for simplicity & M_4 spin]

$\Lambda = H^2(M_4, \Lambda_T)$ even lattice

h: $\Lambda \otimes \mathbb{R} = H^2(M_4, \mathbb{Z}) = (\underset{\text{self dual}}{H^2_+} \oplus \underset{\text{ASD}}{H^2_-}) \otimes \mathbb{Z}$ 2-forms

.... M_6 has conformal structure, we use an underlying metric to give a metric on M_4 which we need to make above splitting.

Our theory reduces on Σ to effective theory defined by Λ, h $\quad$ [$T = \mathbb{R}^n/\Lambda_T$]

Now take $M_4 = TN_{x_1,\dots x_k}$ & suppose x_i distinct (to get smooth space).

$$H^2_{L^2}(M_4, \mathbb{Z}) = \underbrace{\mathbb{Z} \oplus \dots\dots \oplus \mathbb{Z}}_{k}$$

Here $H^2_{L^2}(M_4, \mathbb{Z})$ = group of isomorphism classes of line bundles with L^2 harmonic curvature & trivial monodromy at ∞ (lattice in space of L^2 harmonic forms)

- carries natural quadratic form $\int F \wedge F$

— this is advantage of Taub-NUT over the ALE metric!

When $x_1, \dots x_k$ all nearby have $k-1$ harmonic forms supported near this point — just like the $\mathrm{rk} = k-1$ cohomology of the ALE space — & one more form not supported nearby... ie TN_k gives $U(k)$

ALE gives $SU(k)$...

$$M_\Gamma = \mathbb{R} \times TN_{x_1 \ldots x_k}$$

$$\Lambda = H^2(TN, \mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda_T = \underbrace{\Lambda_T \oplus \cdots \oplus \Lambda_T}_{k}$$

(can replace here $\mathbb{R} \times S^1$ by $S^1 \times S^1$ eg)

Frenkel-Kac-Segal construction of integrable LG reps of level 1 (G ADE):
our 2d CFT $(\Lambda^{p,p'}, u)$ reduces for

$p' = 0$, $\Lambda = \Lambda_T$ to level one loop group.

Character of the loop group at level 1 (ie $\mathrm{Tr}\, q^{L_0}$)
(q = modulus of torus, L_0 rotates S^1)
is $$Z_G(q) = \frac{\theta_{\Lambda_T}(q)}{\eta(q)^{\mathrm{rk}\, G}}$$

or rather its k^{th} power for $TN_{x_1 \ldots x_k}$.

[Unless $G = E_8$ there are more than
one reps $\longleftrightarrow$ theta function at each
point $x_1 \ldots x_k$]

$\mathrm{II}B$ on  $\times TN_{x_1 \ldots x_k} \times \begin{cases} \mathbb{R}^4/\mathbb{Z}_k & \text{last time} \\ \text{resolved/deformed } \mathbb{R}^4/\mathbb{Z}_k & \text{today} \end{cases}$

(*) Answer: $\displaystyle\prod_{i=1}^{k} \frac{\theta_{\Lambda_I + \alpha_i}}{\eta(q)^{\mathrm{rk}\, G}}$

On 6-manifold $S^1 \times \mathbb{R} \times TN_{x_1 \ldots x_k}$ have

hamiltonian $H \longleftrightarrow \frac{\partial}{\partial t}$

momentum $P \longleftrightarrow \frac{\partial}{\partial \theta}$

SUSY Q_i $\{Q_i, Q_j\} = H - P$

minus sign comes from preferred orientation

So if $P|\psi\rangle = n|\psi\rangle \Rightarrow H \geq n$ on these states.

BPS states: $H = n \Leftrightarrow |\psi\rangle \in \bigcap \ker Q_i$

So $\mathcal{H}_{BPS}$ is Q cohomology

On $\mathbb{R} \times TN_k$ we identified

$$\mathcal{H} = \bigoplus \left(H_n = H^*_{L^2}(\mathcal{M}_n) \right)$$ $c_2 = n$ part of BPS states

here $\mathcal{M}_n = \mathcal{M}^k_{Inst}(TN_{x_1 \ldots x_k}, G, c_2 = n)$

On $\mathbb{R} \times S^1 \times \mathbb{R}^3$ we identified

$$\mathcal{H} = \bigotimes_{i=1}^{k} R_i$$ tensor of level 1 irreps of LG, graded by $L_0 = \frac{\partial}{\partial \theta}$

$$\prod_{i=1}^{n} \frac{\theta_{\Lambda_T \cdot \alpha_i}}{\eta(q)^{rk\, G}} = \sum q^n \dim \mathcal{H}_n$$

(choice of $\alpha_i \Longleftrightarrow w_2$ of principal bundle)

Why did we get the same answer today:
cohomology of BRST operator, very robust (index even more so), doesn't change in different limits of our picture.
— can compute index in any good approximation to theory

$M_6 \supset S$ two-manifold

$\Rightarrow$ surface operators related to S in 6dim theory – can see them in full string theory.

When we resolve the singularity, i.e. reduce G to T, have a T-gerbe B

$S \rightsquigarrow Hol(B, S) \in T$ holonomy