Zheng Hao - Categorification of Quantum Groups

Note Title 4/29/2008

Plan: I. Sheaf / Floer theory from POV of quantum mechanics

II. Lusztig canonical basis of U_q^- & categorification

III. Nakajima on quiver varieties $\longleftrightarrow$ rep of $\mathfrak{g}$ Kac-Moody

IV. quiver varieties $\rightsquigarrow$ categorifying reps of $U_q\mathfrak{g}$.

I. Classical phase space for a particle in $\mathbb{R}^3$ is $\mathbb{R}^3 \times \mathbb{R}^3$ = momentum $\times$ position $= T^*(P = \text{position})$
$= T^*(M = \text{momentum})$

Quantum phase space $L^2(M) \xleftrightarrow[\text{Fourier}]{} L^2(P)$

More generally particle on M
$\rightsquigarrow$ classical phase space T^*M
$\rightsquigarrow$ quantum phase space $L^2(M)$

For a general symplectic manifold $X \rightsquigarrow$?
need choice of Lagrangian ..
$\rightarrow$ go to sheaf theory

M, P position / momentum: $D(M) \xleftrightarrow{\text{Fourier}} D(P)$
derived categories of sheaves: substitute for $L^2(M)$.

For general symplectic X with open embedding in a cotangent bundle T^*M

$\Rightarrow$ define $D(M)/N$ localization by

$$N = \{ A \in D(M) : \underbrace{char(A)}_{\text{characteristic cycle}} \cap X = \emptyset \}$$

This category depends on the embedding ....

$\cdots\cdots\rightsquigarrow$ stacks of microlocal sheaves...

<u>Floer theory</u>:

Derived Fukaya category:

Nadler-Zaslow $D(M) \xleftrightarrow{\sim} DFuk(T^*M)$

Objects: "A-branes" Lagrangians + local systems,

morphisms defined via Floer homology

Fourier transform:

$$D(M) \simeq DFuk(M \times P) \simeq D(P)$$

natural from this POV: Fukaya category independent of choice of polarization. (roughly...)

$$D(M)/N \xleftrightarrow{?} DFuk(X)$$

II. Γ finite graph without

$\rightsquigarrow$ generalized Cartan matrix (a_{ij})

$$a_{ij} = \begin{cases} 2 & i=j \\ -\#\text{edges } i \leftrightarrow j & i \neq j \end{cases}$$

e.g. $\underset{A_2}{\bullet\!\!-\!\!\!-\!\!\bullet}$ $\quad (a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Assume (a_{ij}) pos. definite for simplicity,
($\Longleftrightarrow \Gamma$ is an ADE Dynkin diagram), & <u>connected</u>

$\Gamma \rightsquigarrow$ Kac-Moody algebra $\mathfrak{g} = \langle e_i, f_i, h_i \rangle$
& quantum group $U_q = \langle E_i, F_i, K_i \rangle$

$$U_q = \underset{E_i}{U_q^+} \otimes \underset{K_i}{U_q^0} \otimes \underset{F_i}{U_q^-}$$

I = vertices of Γ, H = arrows $\quad \bullet \rightleftarrows \bullet \rightleftarrows \bullet$

$\Omega \subset H$ orientation : choice of "half" of arrows

$V = (V_i)_I$ vector spaces on vertices

$$E_{V,H} = \bigoplus_{h \in H} \mathrm{Hom}(V_{s(h)}, V_{t(h)})$$ representation variety

$$\cup$$

$$E_{V,\Omega} = \bigoplus_{h \in \Omega} \mathrm{Hom}(V_{s(h)}, V_{t(h)})$$

$E_{V,H} = T^* E_{V,\Omega}$. $D(E_{V,\Omega_1}) \underset{\text{Fourier transform}}{\cong} D(E_{V,\Omega_2})$

for any Ω_1, Ω_2

Quiver moduli: $G_V = \prod_{i \in I} GL(V_i) \curvearrowright E_{V,\Omega}$

$\rightsquigarrow$ pass to quotient stack

$X_V = [E_{V,\Omega} / G_V]$

$\rightsquigarrow$ $D(X_V)$ equivariant derived category

Take sum over all dimension vectors

$$\bigoplus_{\dim V} D(X_V)$$

Convolution product: $0 \to T \to V \to W \to 0$

$$\mathrm{Ind}_{T,W}^{V} : D(X_T) \times D(X_W) \longrightarrow D(X_V)$$

$$A \qquad B \longmapsto A * B$$

given by correspondence

$$\begin{array}{ccccc} X_T \times X_W & \xleftarrow{\pi} & Y^V_{T,W} & \xrightarrow{\iota} & X_V \\ A \boxtimes B & & \parallel & & A * B \\ & & [F/G] & & \end{array}$$

where $F = \{ (x_h) \in X_V \ : \ T \text{ is stable under } (x_h) \}$

$G = \{ (g_i) \in G_V \ : \ \text{" } \quad \text{" } \quad \text{" } \quad \text{" } \quad (g_i) \}$

Q = full subcategory of $\bigoplus_V D(X_V)$

formed by direct sums $\bigoplus A_k[n_k]$,

A_k simple perverse sheaves.

<u>Theorem</u> (Lusztig) $(K_0(Q), *) \simeq U_q^-$

algebra isomorphism

(where we work over $\overline{\mathbb{F}_q}$)

$\{$ simple perverse sheaves $\} \longleftrightarrow$ <u>canonical basis</u> of U_q^-.

[Relation with ring $R(\nu)$ of Khovanov - Lauda

Conjecture (Khovanov-Lauda) Ring $R(v)$
is Morita equivalent to $\mathrm{Ext}(L_v, L_v)$
$L_v = \bigoplus$ simple perverse sheaves in $D(X_v)$]

Motivational
Floer version $D(X_v) \simeq DFuk(T^* X_v)$

$T^* X_v$ is independent of choice of Ω

- Singular Lagrangian $\mathcal{L} \subset T^* X_V$: descended from
  $\tilde{\mathcal{L}} \subset E_{V,H} = T^* E_{V,\Omega}$ Union of conormals to G_v orbits
  (Lusztig's quiver variety)
- Smooth Lagrangian $\mathcal{H}^v_{T,W} \subset T^* X_T \times T^* X_W \times T^* X_V$

conormal bundle to $Y^v_{T,W} \subset X_T \times X_W \times X_V$

$\Rightarrow$ category $Q :=$ full subcategory of

$\bigoplus_v DFuk(T^* X_v)$ formed by $\bigoplus A_t [n_t]$

where A_i are the "standard branes" associated each irreducible component of $\mathcal{L}_V$

$$\mathrm{Ind}^V_{T,W} : DFuk(T^*X_T) \times DFuk(T^*X_W) \longrightarrow DFuk(T^*X_V)$$

$$(A, B) \longmapsto \Phi_{\mathcal{L}^V_{T,W}} (A, B)$$

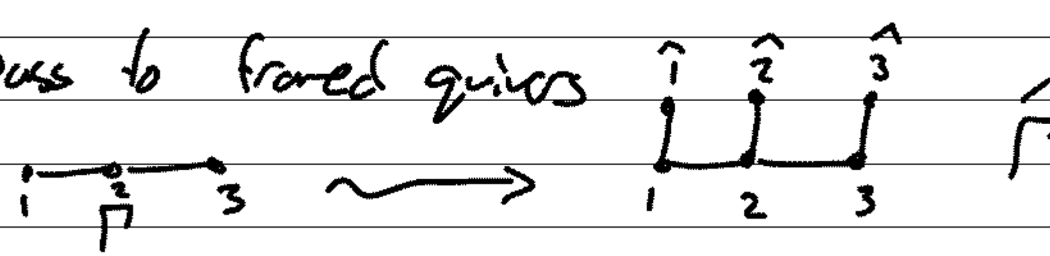
functors defined by Lagrangian correspondences.

<u>Conjecture</u> $(Q, \mathrm{Ind}^V_{T,W})$ in the Fiber picture categorifies U_q^-.
---- manifestly independent of Ω.

III. Categorifying representations

$\Lambda = \Lambda_1 \otimes \dots \otimes \Lambda_n$ tensor product of finite dimensional irreducible $\mathfrak{g}$-modules:

pass to framed quivers



$(\hat{I}, \hat{H})$ $\qquad$ $(\hat{I}, \Omega)$

$\rightsquigarrow$ construct $E_{\hat{V}, \hat{H}}$, $G_V = \prod_{i \in I} GL(V_i)$ stays same

- $T^*[E_{\hat{V},\Omega} / G_V] \supset \mathcal{M}_{\hat{V}}$ a variety, independent of Ω : Nakajima quiver variety
- $\mathcal{H}_{\hat{V},i} \subset \mathcal{M}_{\hat{V}} \times \mathcal{M}_{\hat{V}'}$
- Smooth Lagrangian submanifold

$\hat{V}'$: increase dimension of one vector space V_i by one

- $\mathcal{L}_{\hat{V}} \subset \mathcal{M}_{\hat{V}}$ singular Lagrangian, depends on representation Λ

$$f_i : H^{\text{Borel Moore}}_{tor}(\mathcal{L}_{\hat{V}}) \longrightarrow H^{BM}_{tor}(\mathcal{L}_{\hat{V}'})$$

$$\xi \longmapsto (\xi \times \mathcal{M}_{\hat{V}'}) \cdot \mathcal{H}_{\hat{V},i}$$

intersection with Hecke correspondence in $\mathcal{M}_{\hat{V}} \times \mathcal{M}_{\hat{V}'}$

Similarly define e_i in reverse direction & grading operator h_i

Theorem (Nakajima)

$\{ \bigoplus_{v} H_{top}(\mathcal{L}_{\hat{v}}), e_i, f_i, h_i \}$ realizes the $\mathfrak{g}$-module Λ.

IV [Floer version $\Lambda = \Lambda_1 \otimes \dots \otimes \Lambda_n$ U_q-modules define Q, $\mathcal{E}_i$, $\mathcal{F}_i$, $\mathcal{K}_i$ ...]

Sheaf version:

define microlocalized version

$$Q \subset \bigoplus_{\hat{v}} D([E_{\hat{v},\Omega}/G_v]) / N_{\hat{v}}$$

$$N_{\hat{v}} = \{ A : \text{char } A \cap m_{\hat{v}} = \emptyset \}$$

Q = full subcategory formed by $\bigoplus_k A_k [n_k]$

A_t = simple perverse sheaf
with $\operatorname{Char} A_t \cap \mathfrak{M}_{\hat{V}} \subset \mathcal{L}_{\hat{V}}$

Functors $\mathcal{F}_i : D([E_{\hat{V},\Omega}/G_V])/N_{\hat{V}}$

$$\downarrow$$

$$D([E_{\hat{V}',\Omega}/G_{V'}])/N_{\hat{V}'}$$

given by $X_{\hat{V}} \xleftarrow{\alpha} Y_{\hat{V},i} \xrightarrow{\beta} X_{\hat{V}'}$

$\mathcal{F}_i(A) = \beta_! \alpha^* A$

$Y_{\hat{V},i} \subset X_{\hat{V}} \times X_{\hat{V}'}$ is a subset set
its conormal bundle is $\mathfrak{P}_{\hat{V},i}$.

<u>Theorem</u> $(Q, \mathcal{E}_i, \mathcal{F}_i, K_i)$ categorifies
the U_q-module Λ.

$\leadsto$ find functor isomorphisms for each relation
in U_q.