Xinwen Zhu - H^3 & gerbal extensions

Note Title 4/17/2008

(w/ E. Frenkel)

Goal: towards representation theory of double loop algebras & role of H^3

G reductive$/\mathbb{C}$, $\mathfrak{g} = \mathrm{Lie}\, G$

$K = \mathbb{C}((t)) \supset \mathbb{C}[[t]] = \mathcal{O}$ — Tate vector space & lattice

<u>Def</u>: $\mathfrak{gl}_\infty = \mathrm{End}_{cont} K \supset \mathrm{End}_{cont}\, \mathcal{O} = \mathfrak{gl}_\infty^+$

$GL_\infty = \mathfrak{gl}_\infty^\times$

Embed $G((t)) \hookrightarrow GL_\infty$. To construct central extensions of LHS pull back from central extensions of RHS

Choose a projection $\pi: K \to \mathcal{O}$

$\Rightarrow$ linear map $\pi: \mathfrak{gl}_\infty \to \mathfrak{gl}_\infty^+$

$\mathfrak{gl} \subset \mathfrak{gl}_\infty^+$ consisting of $f: \mathcal{O} \to \mathcal{O}$ s.t. $f|_{\text{open subspace}} = 0$

$\Rightarrow$ Define $\widetilde{\mathfrak{gl}}_\infty = \{(a,g) \in \mathfrak{gl}_\infty^+ \times \mathfrak{gl}_\infty : a - \pi(g) \in \mathfrak{gl}\}$

→ short exact sequence of Lie algebras

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{gl} & \to & \widetilde{\mathfrak{gl}}_\infty & \xrightarrow{\varphi} & \mathfrak{gl}_\infty & \to & 0 \\ & & \downarrow{\scriptstyle Tr} & & & & \| & & \\ 0 & \to & \mathbb{C} & \to & \widehat{\mathfrak{gl}}_\infty & \to & \mathfrak{gl}_\infty & \to & 0 \end{array}$$

pushout along trace map on $\mathfrak{gl}$ to get central extension of $\widehat{\mathfrak{gl}}_\infty$.

Group counterpart:

$$GL_\infty^+ = (\mathfrak{gl}_\infty^+)^\times, \quad \widetilde{GL}_\infty = \mathfrak{gl}_\infty^\times$$

$GL \subset GL_\infty^+$: identity on open subspace of $\mathcal{O}$.

$$\Rightarrow \quad 1 \to GL \to \widetilde{GL}_\infty \to GL_\infty \xrightarrow[\deg]{} \mathbb{Z} \to 0$$

where $\deg : GL_\infty \to \mathbb{Z}$

$$\deg g = \dim \frac{\mathcal{O}}{\mathcal{O} \cap g\mathcal{O}} - \dim \frac{g\mathcal{O}}{\mathcal{O} \cap g\mathcal{O}}$$

So if $GL_\infty^0 = \ker(d_g) \to$

$$\begin{array}{ccccccccc} 1 & \to & GL & \to & \widetilde{GL}_\infty & \to & GL_\infty^0 & \to & 1 \\ & & \downarrow \det & & & & \| & & \\ 1 & \to & \mathbb{C}^* & \to & \widehat{GL}_\infty^0 & \to & GL_\infty^0 & \to & 1 \end{array}$$

<u>Lemma</u> $\exists ! /\sim$ central extension of GL_∞ whose restriction to GL_∞ is as above.

$\det \in H^1(GL, \mathbb{C}^*) \to$ transgress to class in H^2 ..

<u>Construction of representations</u>

Spin module / fermionic Fock space:

$K^* = \mathbb{C}((t))\,dt$

Take Clifford algebra $Cl_{K \oplus K^*}$ associated to $(K \oplus K^*, \langle\,\rangle)$.

CM_K := discrete modules for $Cl_{K\oplus K^*}$

Standard object : $\Pi_{\mathcal{O}} = \mathrm{Ind}_{\Lambda^\bullet(\mathcal{O}+\mathcal{O}^\perp)}^{Cl(K\oplus K^*)} \mathbb{C}|0\rangle$

– associated to Lagrangian $\mathcal{O} + \mathcal{O}^\perp \subset K \oplus K^*$

<u>Lemma</u> CM_K is semisimple & all simple objects are isomorphic to $\Pi_{\mathcal{O}}$.

$GL_\infty \curvearrowright Cl_{K\oplus K^*}$ $\forall g \in GL_\infty$, the twist $\Pi_{\mathcal{O}}^g \simeq \Pi_{\mathcal{O}}$. Choosing such isomorphisms get a projective action of GL_∞ on $\Pi_{\mathcal{O}}$.

⇝ hence a central extension acts.

<u>Lemma</u> $\widehat{GL_\infty}$ is the central extension defined by this projective representation.

Double Loop Groups $G((t))((s))$, $\mathfrak{g}((t))((s))$

Universal central extension of a Lie algebra $\mathfrak{g} \otimes A$ has the form [$\mathfrak{g}$ simple]

$$0 \to \frac{\Omega^1 A}{dA} \to \widehat{\mathfrak{g} \otimes A} \to \mathfrak{g} \otimes A \to 0$$

Huge in the double loop case.

But $H^3(\mathfrak{g}((t))((s)))$ fin dim, usually isomorphic (eg so_n) to $HC_2(\mathbb{C}((t))((s)))$

$\mathfrak{g} = sl_n$: meaningful 3-cocycle

$$c(\xi_1, \xi_2, \xi_3) = \operatorname{Res}\operatorname{Res}(\operatorname{Tr}(\xi_1 d\xi_2 d\xi_3))$$

Let $\mathcal{C}$ be an abelian category

G action on $\mathcal{C}$:

$$g \in G \longmapsto F_g : \mathcal{C} \xrightarrow{\sim} \mathcal{C} \quad \mathbb{C}\text{-linear additive}$$

$$g, g' \in G \rightsquigarrow c(g, g') : F_g F_{g'} \xrightarrow{\sim} F_{gg'}$$

with compatibility for

$$\begin{array}{ccccc} & & F_{gg'}F_{g''} & & \\ & \nearrow & \circlearrowright & \searrow & \\ F_g F_{g'} F_{g''} & & & & F_{gg'g''} \\ & \searrow & & \nearrow & \\ & & F_g F_{g'g''} & & \end{array}$$

A <u>gerbal action</u> of $G \curvearrowright \mathcal{C}$ consists of

- $g \in G \rightsquigarrow F_g : \mathcal{C} \circlearrowleft$

  $F_e \simeq \mathrm{Id}$, $F_g F_{g'} \simeq F_{gg'}$ but don't fix the isomorphisms

Making choices of these isos get a cocycle

$$\begin{array}{ccccc} & \nearrow & F_{gg'}F_{g''} & \longrightarrow & F_{gg'g''} \\ F_g F_{g'} F_{g''} & & & & \Big\downarrow \; a(g,g',g'') \in Z(\mathcal{C})^{\times} \\ & \searrow & F_g F_{g'g''} & \longrightarrow & F_{gg'g''} \end{array}$$

$Z(\mathcal{C}) = \mathrm{End}_{\mathcal{C}}\, 1$

Claim $G \curvearrowright \mathcal{C}$ gerbal $\Rightarrow$

i. G acts on $Z(\mathcal{C})$

ii. well defined class $a \in H^3(G, Z(\mathcal{C})^\times)$

& $a = 0$ iff the action can be upgraded to an honest action

& choices form a $H^2(G, Z(\mathcal{C})^\times)$-torsor

Remark $\mathbb{C} \subset Z(\mathcal{C})$ with trivial G-action

Examples i) $G \curvearrowright A$ algebra $\Rightarrow$ honest action $G \curvearrowright A$-mod.

ii) G acts on A by outer automorphisms $\Rightarrow$ gerbal action $G \curvearrowright A$-mod

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Def A 2-group is a monoidal groupoid $\mathcal{G}$ s.t. $\pi_0 \mathcal{G}$ is a group.

Denote $\pi_1(\mathcal{G}) = \mathrm{End}_{\mathcal{G}} I$

<u>Ex.</u>
- Any group is a 2-group with trivial π_1
- $B\mathbb{G}_m$ is a 2-group
- Any Picard groupoid is a 2-group (commutative)
- $GL(\mathcal{C}) = \begin{cases} \text{Obj} = \text{auto equivalences} \\ \text{Mor} = \text{isoms of functors} \end{cases}$

  $\mathcal{C}$ abelian category

An action of G on $\mathcal{C}$ is a 2-group homomorphism $G \longrightarrow GL(\mathcal{C})$

A gerbal action is $G \longrightarrow \pi_0(GL(\mathcal{C}))$

<u>Def/prop.</u> $GL(\mathcal{C})(A) = GL_A(\mathcal{C}_A)$

$A/\mathbb{C}$ commutative

$GL(\mathcal{C})$ is a sheaf of 2-groups in $(\mathrm{Aff}/\mathbb{C})_{fppf}$

Def Let $\mathfrak{g}$ be a Lie algebra. A gerbal action of $\mathfrak{g}$ on $\mathcal{C}$ is a group homomorphism
$\hat{G} \longrightarrow \pi_0(GL(\mathcal{C}))$ ($\hat{G}$ · formal group)

$\rightsquigarrow$ as $H^3(\mathfrak{g}, Z(\mathcal{C}))$

If $G \circlearrowright \mathcal{C}$ gerbal $\Rightarrow$
get an honest action of the 2-group
$\hat{G} = \begin{cases} \text{Ob}: & g \in G \\ \text{Mor}: & Z(\mathcal{C})^* \end{cases}$ $(g \otimes g') \otimes g'' \xrightarrow{a} g \otimes (g' \otimes g'')$

<u>Double loop groups</u>

Let $GL_{\infty,\infty} = GL_\infty(\mathfrak{gl}_\infty) \circlearrowright \mathcal{U} = \mathbb{C}((t))((s))$ (as assoc. algebra)

$\mathfrak{gl}_{\infty,\infty} = \mathfrak{gl}_\infty(\mathfrak{gl}_\infty)$

Goal: construct abelian categories on which $GL_{\infty,\infty}$ acts, realizing nontrivial third cohomology class

First attempt: $Cl_{V\oplus V^*}$-mod $\circlearrowleft Gl_{\infty,\infty}$ honest action.

Let $V = \mathbb{C}((t))[s]$, $V^* = s^{-1}\mathbb{C}((t))[s^{-1}]$

restricted dual.

$L \subset V$ is a lattice if $L / L \cap s^n V$ is a lattice in $V/s^n V \cong \mathbb{C}((t))$

Def $\mathcal{C}_V$ = subcategory of $Cl_{V\oplus V^*}$-mod consisting of $\pi_L = \mathrm{Ind}_{\Lambda(L\oplus L')}^{Cl_{V\oplus V^*}}(\mathbb{C}|0\rangle)$ & their direct sums.

Theorem $Gl_{\infty,\infty}$ acts on $\mathcal{C}_V$ gerbally, giving a nontrivial class $\eta \in H^3(Gl_{\infty,\infty}, \mathbb{C}^*)$. The restriction of this action to $Gl_{\infty,\infty}^+ := Gl_\infty^+$ (as before) can be canonically upgraded to an honest action.

Recall

$$1 \to GL(A) \to \widetilde{GL_\infty}(A) \to GL_\infty(A) \xrightarrow{\deg} K_0(A) \to 0$$

A any ring. Cokernel is $K_0(A)$:

Let $A = gl_\infty$. $K_0(gl_\infty) = 0 \implies$

$$1 \to GL(gl_\infty) \to \widetilde{GL}_\infty(gl_\infty) \to GL_{\infty,\infty} \to 1$$

There is a central extension of $GL(gl_\infty)$ by $\mathbb{C}^*$ constructed just as in the beginning of the talk for GL_∞.

$\to$ class $\xi \in H^2(GL(gl_\infty), \mathbb{C}^*)$

coming from det by transgression.

<u>Theorem</u> η is the transgression of ξ.