A. Braverman – Double affine Grassmannian

Note Title 11/29/2007

via gauge theory on ALE spaces

Dream: Create Langlands duality picture for affine Kac-Moody groups

Langlands duality

"usual" automorphic forms: classical Satake w/ D. Kazhdan

arXiv 0711.2083 Geometric Duality on $Bun_G X$ – geometric Satake. w/ M. Finkelberg

1. Review of usual geometric Satake

G reductive / $\mathbb{C}$. $K = \mathbb{C}((s))$ $\cup$ $\mathcal{O} = \mathbb{C}[[s]]$

$Gr_G = G(K)/G(\mathcal{O})$ ind scheme.

Λ = coweights of G

$G(\mathcal{O})$ orbits on $Gr \longleftrightarrow \lambda \in \Lambda^+$ dominant coweights

$Gr_G^\lambda = G(\mathcal{O}) s^\lambda G(\mathcal{O})$ fin dimensional

$\lambda: \mathbb{G}_m \to T \subset G$

$Gr = \bigcup \overline{Gr^\lambda_G}$ singular projective varieties

$\mathcal{P} = \mathrm{Perv}_{G(\mathcal{O})} \mathcal{G}r_G$: semisimple category, has a tensor structure

<u>Theorem</u>

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\sim"] \arrow[dr] & \mathrm{Rep}\, G^\vee \arrow[d, "\text{forget}"] \\ & \mathrm{Vect} \end{tikzcd}$$

canonical equivalence of tensor functors

$S \mapsto H^*(Gr, S)$

Note $\overline{Gr}^\lambda = \bigcup_{\mu \le \lambda} Gr^\mu$

IC^λ : intersection cohomology sheaf of $\overline{Gr}^\lambda$

<u>Corollary</u> The stalk of IC^λ at a point of Gr^μ is equal to the q-analog of the weight multiplicity $L(\lambda)_\mu$

[known before Theorem!]

$L(\lambda)$ = irred $G^\vee$ module with h.w. λ

$L(\lambda)_\mu$ has a filtration (Brylinski–Kostant):

$e \in \mathfrak{g}^\vee$ regular nilpotent (e.g. $\sum_i e_i$ simple root gens.)

--- get filtration induced by action of e.

The stalk $IC^{\lambda}_{\mu} \simeq Gr^F(L(\lambda)_\mu)$

associated graded (roughly speaking)

Explicit model for singularities:
$\exists$ canonical transversal slice $\overline{W}^{\lambda}_{\mu}$ to
$Gr^{\mu} \subset \overline{Gr^{\lambda}}$. Let $W^{\lambda}_{\mu} = \overline{W}^{\lambda}_{\mu} \cap Gr^{\lambda}$
hyperkähler. (Mirković) affine conical variety
Sl_n - these are quiver varieties.

Goal: explain analogs of $\overline{W}^{\lambda}_{\mu}$ in the affine case.
$IC(\overline{W}^{\lambda}_{\mu})$ = q-analogs of weight multiplicities.

<u>Speculation</u> Want to replace everywhere
G by affine Kac-Moody group.
Assume G simply connected & any simple.

$$1 \to \mathbb{G}_m \to \hat{G} \to G[t,t^{-1}] \to 1$$

Central extension. $\mathbb{G}_m \hookrightarrow \hat{G}$ by loop rotation

$G_{aff} = \hat{G} \rtimes \mathbb{G}_m$. Consider as abstract group -- has Lie algebra $\mathfrak{g}_{aff}$, affine Kac-Moody Lie algebra $\rightsquigarrow$ Dynkin diagram

Geometric Satake:

geometry of $Gr_G \longleftrightarrow$ reps of $G^\vee$

has obvious affine analog?

$G^\vee_{aff}$ dual affine group, has Lie algebra $\mathfrak{g}^\vee_{aff}$ with dual Dynkin diagram.

Warning: in general $G^\vee_{aff} \neq (G^\vee)_{aff}$.

G not simply laced $\rightsquigarrow$ <u>twisted</u> affine K-M algebras.

$\mathfrak{g}^\vee_{aff} \supset \mathfrak{g}^\vee$ should integrate to $G^\vee_{aff} \supset G^\vee$

-- uniquely characterizes $G^\vee_{aff}$.

Central extension interchanged with loop rotation.

Rep $G^\vee_{aff}$:= integrable representation of $\mathfrak{g}^\vee_{aff}$ on which the adjoint group $G^\vee$ acts.

Usual $\otimes$ makes sense ... have to allow some infinite direct sums — keep track of gradings wrt loop rotations.

Level k rep : action of central $\mathbb{G}_m \subset G^\vee_{aff}$

$(k>0)$ Irreps of $G^\vee_{aff}$ of level k

$\Longleftrightarrow \Lambda^+_k \times \mathbb{Z}$: $\Lambda^+_k \subset \Lambda^+$ k-Weyl alcove.

The Brylinski-Kostant filtration on $L(\lambda)_\mu$ also makes sense

$e_{aff} = \sum e_i$ sum over all vertices of Dynkin diagram.

$\rightsquigarrow gr^F L(\lambda)_\mu$ makes sense $\rightsquigarrow$ q-analog of weight multiplicities.

Defining the affine Grassmannian:

$K = \mathbb{C}[s, s^{-1}] \supset \mathcal{O} = \mathbb{C}[s]$

$$G_{aff}(K) / G_{aff}(\mathcal{O}) = Gr_{G_{aff}} \longrightarrow \mathbb{Z}$$

$$Gr_{G_{aff},k} = \pi^{-1}(k) \quad k > 0 \qquad \qquad \mathbb{Z} = Gr_{\mathbb{G}_m}$$

<u>Lemma</u> $G_{aff}(\mathcal{O})$ orbits on $Gr_{G_{aff},k}$ are in bijection with $\Lambda^+_{k,aff}$

Not clear how to think of these algebro-geometrically: very infinite dimensional.

$\rightsquigarrow$ try to construct finite dimensional models / transversal slices

Not clear what above relation

Give answer, motivation come afterwards.

<u>History</u>: Nakajima: G simply laced $\rightsquigarrow \Gamma \subset SL_2\mathbb{C}$. <u>Integrable</u> reps of $\mathfrak{g}_{aff}$ of level n realized in geometry of moduli of rank n vector bundles on $\mathbb{C}^2/\Gamma$

I. Frenkel: should have a dual construction,
integrable reps of G_{aff} of level k
$\longleftrightarrow$ moduli of G-bundles on $\mathbb{C}^2/\mathbb{Z}_k$

find Int. reps of $\underline{G^\vee_{aff}}$ of level k via
G-bundles on $\mathbb{C}^2/(\mathbb{Z}_k =: \Gamma_k)$

<u>Def</u> $\mathrm{Bun}^a_G(\mathbb{C}^2) :=$ G-bundles on $\mathbb{P}^2$
trivialized at $\mathbb{P}^1_\infty \subset \mathbb{P}^2$,
$c_2 = a \geq 0$

smooth quasi affine alg. variety

Affinization: $\mathrm{Bun}^a_G \subset \mathcal{U}^a_G(\mathbb{C}^2)$ Uhlenbeck space

set theoretically: $= \coprod_{0 \leq k \leq a} \mathrm{Bun}^k_G \mathbb{C}^2 \times \mathrm{Sym}^{a-b} \mathbb{C}^2$

Γ_k acts on $\mathrm{Bun}^a_G(\mathbb{C}^2)$ via action on $\mathbb{C}^2$
& homomorphism $\bar{\mu} : \Gamma_k \longrightarrow G$
... gives action on framings
<u>Note</u>: set of $\bar{\mu} : \Gamma_k \to G \Longleftrightarrow \Lambda^+_k$.

$\rightsquigarrow$ Defines an action of Γ_k on $Bun_G^a(\mathbb{C}^2)$

$Bun_{\bar{\mu}}^a$ --- fixed points

$\bar{\lambda} : \Gamma_k \longrightarrow G$ action of Γ_k on fiber at 0 (the fixed bundles are Γ_k equivariant so 0-fiber carries Γ_k action)

$\Rightarrow$ space $Bun_{\bar{\mu}}^{\bar{\lambda},a}(\mathbb{C}^2) \subset U_{\bar{\mu}}^{\bar{\lambda},a}$ closure in $U^a(\mathbb{C}^2)$

known for sl_n

<u>Conjecture!</u> 1. $Bun_{\bar{\mu}}^{\bar{\lambda},a}$ is connected if not empty

2. $Bun_{\bar{\mu}}^{\bar{\lambda},a} \neq \emptyset$ if $\exists \lambda\mu \in \Lambda_{k,aff}^{+}$

s.t. $\lambda = (k, \bar{\lambda}, m)$ & $\lambda \geq \mu$
$\mu = (k, \bar{\mu}, l)$
with $a = k(m-l) + \langle \bar{\lambda}, \bar{\lambda} \rangle - \langle \bar{\mu}, \bar{\mu} \rangle$

... ie space is parametrized by pairs of dominant weights of dual affine algebra with $\lambda \geq \mu$.

.... True for Sl_n (Nakajima),
any G for $k=1$.

Idea! $\mathcal{U}^\lambda_\mu$ is the affine analog of $\overline{W}^\lambda_\mu$

affine conical variety Bun^λ_μ " " " " " W^λ_μ

<u>Conjecture 2</u> $IC(\mathcal{U}^\lambda_\mu)_0 \cong gr^F L(\lambda)_\mu$

- Known for $k=1$ (level 1) from Braverman-Gaitsgory-Finkelberg
- Sl_n: Conjecture 1 known, Conjecture 2 known if grading disregarded (Nakajima)
- $k >> 0$ known: spaces <u>become</u> the $\overline{W}^\lambda_\mu$ ...

$$(Bun^a_G(\mathbb{C}^2))^{\Gamma_k} = (Bun^a_G)^{\mathbb{C}^*} \quad k>>0$$

$\bar\lambda, \bar\mu \in \Lambda^+$ for $k>>0$

$$Bun^{a\,\bar\lambda}_{\bar\mu} = \begin{cases} \phi & \bar\lambda \not\geq \bar\mu \quad a = \langle\bar\lambda, \iota\rangle - \langle\bar\mu, a\rangle \\ W^{\bar\lambda}_{\bar\mu} & \bar\lambda \geq \bar\mu \end{cases}$$