D. Gaitsgory - Localization & Reps of Affine Algebras

Note Title 11/26/2007

w/ E. Frenkel

$\mathfrak{g} / \mathbb{C} \rightsquigarrow$ affine Kac-Moody

$$0 \to \mathbb{C} \to \hat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) \to 0$$

$\hat{\mathfrak{g}}_\kappa$-mod : reps where center acts as 1.

Today κ = critical = $-\frac{1}{2}$ Killing

Want to describe these categories as
D-modules on homogeneous spaces or as
$\mathcal{O}$-modules on parameter spaces

Finite dim case:

$$\begin{tikzcd} D\text{-mod}(G/B) \arrow[r] \arrow[dr, "\Gamma"] & \mathfrak{g}\text{-mod} \\ & \mathfrak{g}\text{-mod}_0 \arrow[u, hook] \end{tikzcd}$$

center acts as on trivial

<u>Theorem</u> (Beilinson-Bernstein)
Γ is an exact equivalence of abelian categories

For us $G/B \rightsquigarrow Fl_G = G((t))/I$
strict ind scheme of ind finite type.

$$D\text{-mod}(Fl_G)_\kappa \xrightarrow{\Gamma} \hat{\mathfrak{g}}_\kappa\text{-mod}$$

Can't be an equivalence: RHS much bigger

$\rightsquigarrow$ want to impose conditions on center to get close to an equivalence.
So we assume κ = critical $\rightsquigarrow$
get a large center for the category

$$Z(\hat{U}(\hat{\mathfrak{g}}_{crit})) \underset{\text{Feigin-Frenkel}}{\cong} \mathrm{Fun}(Op_{G^\vee}(D^\times))$$

topological commutative algebra

opers on punctured disc for dual group

$$\begin{array}{ccc} Op_{G^\vee}(D^\times) & \cong & \mathbb{C}((t))^r \quad \text{non canonically} \\ \downarrow & & \\ Loc_{G^\vee}(D^\times) & & \end{array}$$

but opers are a (ind) scheme not just stack

$$\begin{array}{ccccc} Op^{reg}_{G^\vee} & \subset & Op^{nilp}_{G^\vee} & \subset & Op_{G^\vee}(D^\times) \\ \parallel & & \parallel & & \parallel \\ \mathbb{C}[[t]]^r & \subset & \prod_{i=1,\dots,r} \mathbb{C}[[t]]t^{-d_i} & \subset & \mathbb{C}((t))^r \end{array}$$

$Op^{nilp}_{G^\vee} \longrightarrow$ local systems with irreg singularity & nilpotent monodromy.

$\hat{\mathfrak{g}}_{crit}\text{-mod}^{nilp} \subseteq \mathfrak{g}_{crit}\text{-mod}$:

subcategory where center acts through quotient
$\mathrm{Fun}(\mathrm{Op}_{G^\vee}^{nilp})$.

Can check $D\text{-mod}(Fl_G)_{crit} \xrightarrow{\Gamma} \hat{\mathfrak{g}}_{crit}\text{-mod}^{nilp}$

Still not an equivalence, but much closer!

Problem: Γ is not exact in an essential way.
$\rightsquigarrow$ pass to derived categories

$D(D\text{-mod}(Fl_G)_{crit}) \longrightarrow D(\hat{\mathfrak{g}}_{crit}\text{-mod}^{nilp})$
not usual derived category: need to control behavior at infinity.

LHS carries an interesting strange t-structure, where Γ is conjecturally exact.

Why can't Γ be an equivalence?

$\mathcal{F} \in D\text{-mod}(Fl_G)_{crit} \Rightarrow$
$$\Gamma(Fl_G, \mathcal{F}) \circlearrowleft \mathrm{Fun}(\mathrm{Op}_{G^\vee}^{nilp})$$
but $D\text{-mod}(Fl)$ itself has no center
- we've gained too many endomorphisms!

So modify LHS so it gains a center as well & account for lack of full-faithfulness.

Use Arkhipov-Bezrukavnikov :
[suppress twist in notation, doesn't affect category itself]

$D(\mathcal{D}\text{-mod}(Fl_G))$ carries action of the monoidal category
$D(QCoh\ \tilde{N}^\vee/G^\vee)$ $\quad \tilde{N}^\vee = T^*(G^\vee/B^\vee)$

But $Op_{G^\vee}^{nilp} \xrightarrow{Res} \tilde{N}^\vee/G^\vee$
map given by taking residue of the nilpotent connection & associated flag.

Compatibility: $\mathcal{F} \in \mathcal{D}(Fl_G)$, $M \in QC(\tilde{N}/G^\vee)$

$$(*)\quad \Gamma(Fl_G, \mathcal{F} * M) = \Gamma(Fl_G, \mathcal{F}) \otimes_{Fun(Op^{nilp})} Res^*(M)$$

<u>Analogy</u> $\quad$ $V_1 \xrightarrow{\phi_V} V_2$ $\quad$ vector spaces
$A_1 \circlearrowright V_1$, $A_2 \circlearrowright V_2$
$A_1 \longrightarrow A_2$ $\quad$ assoc. algebras

$\Rightarrow$ map of A_2-modules

$$A_2 \underset{A_1}{\otimes} V_1 \longrightarrow V_2$$

Apply categorical version: V_1, V_2 triangulated, A_1, A_2 monoidal triangulated categories

$\rightsquigarrow$ can't quite do this $\otimes$ product

But our categories are from dg categories or other homotopical replacement.

... will ignore these subtleties

$$\begin{tikzcd} V_1 \arrow[r] & V_2 \\ A_1 \arrow[u, bend left] \arrow[r] & A_2 \arrow[u, bend right] \end{tikzcd} \Rightarrow A_2 \underset{A_1}{\otimes} V_1 \longrightarrow V_2$$

In our case $V_1 = D(\mathcal{D}\text{-mod}(Fl_G))$

$V_2 = D(\hat{\mathfrak{g}}_{crit}\text{-mod}^{nilp})$

$A_1 = D(QCoh(\tilde{\mathcal{N}}^\vee/G^\vee))$

$A_2 = D(QCoh(Op_{G^\vee}^{nilp}))$

so we'll base change V_1 from A_1 to A_2

$$Op_{G^\vee}^{nilp} \underset{\tilde{\mathcal{N}}^\vee/G^\vee}{\times} D(\mathcal{D}(Fl_G)_{crit}) \xrightarrow{\Gamma} D(\hat{\mathfrak{g}}_{crit}\text{-mod})$$

Conjecture Γ_{Op} is an equivalence of categories

(Problem: don't know enough objects on RHS)

Theorem 1 Γ_{Op} is fully faithful

Let $I^0 \subset I$ nilpotent radical.
Can consider full subcategory of I^0-invariant objects

$$\mathcal{O}p^{nilp} \underset{\tilde{\mathcal{N}}^\vee/G}{\times} D(\mathcal{D}(Fl_G))^{I^0} \xrightarrow{\Gamma_{Op}} D(\hat{\mathfrak{g}}_{crit})^{I^0}$$

"category $\mathcal{O}$" type version

Theorem 2 Γ_{Op} on I^0-equivariant categories is an equivalence

.... here use Verma modules to prove.

Different description of RHS

$$\begin{array}{c} \tilde{\mathfrak{g}}^\vee \\ \downarrow \\ \mathfrak{g}^\vee \end{array}$$

Grothendieck alteration: dominant, W-Galois over reg. semisimple locus.

Miura opers:

$$MOp^{nilp}_{G^\vee} = Op^{nilp}_{G^\vee} \times_{\mathfrak{g}^\vee/G^\vee} \tilde{\mathfrak{g}}^\vee / G^\vee$$

(here don't need derived version).

$$St^\vee_{G^\vee} = \tilde{N}^\vee \times_{\mathfrak{g}^\vee} \tilde{\mathfrak{g}}^\vee \Rightarrow$$

$$MOp^{nilp}_{G^\vee} = Op^{nilp}_{G^\vee} \times_{\tilde{N}^\vee/G^\vee} St^\vee_{G^\vee} / G^\vee$$

... add extra flag to our underlying local system.

<u>Theorem 3</u> $D^+(\hat{\mathfrak{g}}_{crit}\text{-mod})^{I^0} \simeq D^+(QCoh\ MOp^{nilp}_{G^\vee})$

Why Miura opers?

$$MOp^{nilp}_{G^\vee} \longleftarrow \bigcup_{w \in W} MOp^{nilp, w}_{G^\vee}$$

decomposition into affine varieties (isomorphism on level of sets).

For each w we've known for long to assign Wakimoto modules

$$QCoh(MOp_{G^\vee}^{nilp,\omega}) \xrightarrow{W} \hat{\mathfrak{g}}_{crit}\text{-mod}^{nilp}$$

Above construction glues these together: Theorem 3 follows immediately by basechanging over Res the theorem of Bezrukavnikov:

$$D(\mathcal{D}mod(Fl_G)^{I_0}) \simeq D(QCoh(St_{G^\vee}/G^\vee))$$

How to prove Theorem 1?

Use affine Grassmannian

$$\begin{array}{c} Fl_G \\ \downarrow \\ Gr_G = G((t))/G[[t]] \end{array}$$

$$D\text{-mod}(Gr_G) \longrightarrow \hat{\mathfrak{g}}_{crit}\text{-mod}^{res}$$

correspond to $Op_p^{res} \subset Op_p^{nilp}$.

Relevant base change here:

$$Op_{G^\vee}^{res} \underset{\cdot/G^\vee}{\times} D\text{-mod}(Gr_G)$$

$(QC(\cdot/G^\vee) \curvearrowright D\text{-mod}(G)$ by geometric Satake)

The resulting functor to $\hat{\mathfrak{g}}_{crit}\text{-mod}^{reg}$ is <u>exact</u>.

<u>Theorem</u> $\quad Op^{reg}_{G^\vee} \times_{/G^\vee} D\text{-mod}(Gr_G) \longrightarrow D(\hat{\mathfrak{g}}_{crit}\text{-mod}^{reg})$

is fully faithful

— results from ability to calculate self Ext of Vacuum module

$\mathbb{V} = \mathrm{Ind}^{\mathfrak{g}((t))}_{\mathfrak{g}[[t]]}(\mathbb{C})$ .. still can't prove

equivalence without the Iwahori invariance.

Want to use this to describe $D\text{-mod}(Fl_G)$

$G^\vee/B^\vee \overset{i}{\hookrightarrow} \tilde{N}^\vee \quad \rightsquigarrow$

$\bullet/B^\vee \hookrightarrow \tilde{N}^\vee/G^\vee$

<u>Theorem 4</u> $\bullet/B^\vee \times_{\tilde{N}^\vee/G^\vee} D(D\text{-mod}(Fl_G))$

$\cong \bullet/B^\vee \times_{\bullet/G^\vee} D(D\text{-mod}(Gr_G))$

$$\begin{tikzcd}
Op_{G^\vee}^{reg} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & Op_{G^\vee}^{nilp} \arrow[d, "Res"] \\
\cdot/B^\vee \arrow[r, hook] & \tilde{N}^\vee/G^\vee
\end{tikzcd}$$

& same holds on level of representations

$$D(\hat{g}^{crit}\text{-mod}^{reg}) \simeq Op_{G^\vee}^{reg} \times_{Op_{G^\vee}^{nilp}} D(\hat{g}_{crit}\text{-mod}^{nilp})$$

How does this give what we want?

$$Op^{nilp} \times_{\tilde{N}^\vee/G^\vee} D(\mathcal{D}(Fl_G)) \xrightarrow{\Gamma_{Fl}} D(\hat{g}_{crit}\text{-mod}^{nilp})$$

$$Op^{reg} \times_{Op^{nilp}} Op^{nilp} \times_{\tilde{N}^\vee/G^\vee} D(\mathcal{D}(Fl_G)) \longrightarrow D(\hat{g}^{crit}\text{-mod}^{reg})$$

$$\shortparallel$$

$$Op^{reg} \times_{\cdot/B^\vee} \cdot/B^\vee \times_{\tilde{N}/G^\vee} D(\mathcal{D}(Fl_G))$$

$$\shortparallel$$

$$Op^{reg} \times_{\cdot/G^\vee} D\text{-mod}(Gr_G) \rightsquigarrow \text{set Thm 4}$$

To bootstrap to flags use Nakayama lemma:
know a map is an isomorphism
when restricted to regular opers
($\Longleftrightarrow$ fully faithfulness) $\Longrightarrow$
implies it remains an isom
on nilpotent opers (must put
gradings on everything, coming from
a coordinate on the formal disc)