S. Gukov - D-branes & Representations

Note Title 11/28/2007

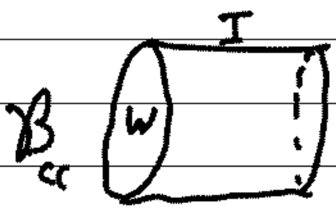
(w/ E. Witten)

Goal: understand admissible $G_{\mathbb{R}}$-reps in terms of D-branes

Notation: G compact Lie group (simple)
$G_{\mathbb{C}}$ complexification of G
$G_{\mathbb{R}}$ real form of $G_{\mathbb{C}}$

Take GL twist of $N=4$ SYM - relates to complexified flat connections.

Start on $M_4 = W \times I$ W 3-manifold, I interval



Put two boundary conditions $\mathcal{B}_{cc}$ & $\mathcal{B}'$ preserving topological SUSY for the 4d TFT

$\mathcal{B}_{cc}$: including condition $F_A^+\big|_{W\times\{0\}} = 0$

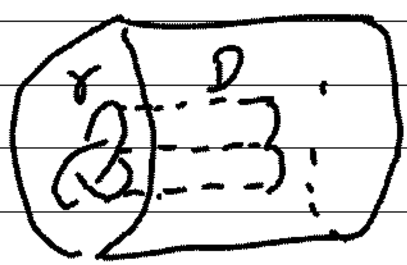
$\mathcal{B}'$: including condition $\phi = 0$

collapse interval :  $\Rightarrow$ 3d TQFT on W

involves only scalar field — all Higgs fields projected out $\rightsquigarrow$ Chern–Simons theory with gauge group G (compact)

Add another ingredient: surface operators in the 4d theory — operators supported on 2d surface $D \subset M_4$

e.g. $D = \gamma \times I$, $\gamma \in W$ curve in the 3-fold



$\Rightarrow$ line operator in the 3d theory supported on $\gamma \in W$:
Wilson line $W_R(\gamma)$ labelled by R rep. of G compact.

<u>Remark</u>: surface operators in the 4d theory are parametrized by data which is very different from representations of G — (true surface operators)

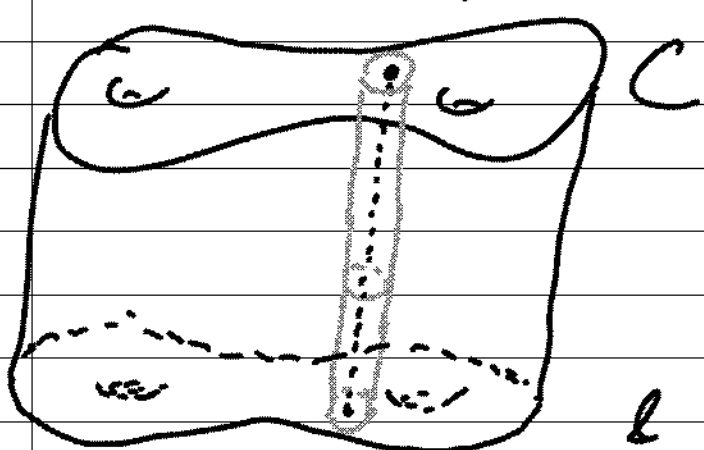
$\longleftrightarrow$ discrete data, + $\alpha, \beta, \gamma, \eta$

We'll see how some of these become discrete & correspond to reps R.

Take $W = \mathbb{R} \times C$ $\quad C$ = Riemann surface

$\mathbb{R}$ = "time".

$\gamma = \mathbb{R} \times \{p\}$



Hamiltonian approach:

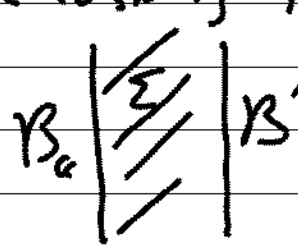
$\Downarrow$

Hilbert space $\mathcal{H}$

Let's replace C by a punctured disc, & $\mathcal{H}$ will be our representation space.

- specify boundary conditions only at the puncture.

In 4d gauge theory: we're considering $M_4 = \Sigma \times C$ where $\Sigma = \mathbb{R} \times I$:



4d gauge theory on $M = \Sigma \times C$

$\Longleftrightarrow$

2d topological σ-model $\Sigma \longrightarrow \mathcal{M}_H(G, C)$

Hitchin moduli space.

This is hyperkähler: complex structures I, J, K

& Kähler 2 forms ω_I, ω_J, ω_K.

For applications to rep. theory take $C = D^*$

punctured disk with surface operator at the

puncture $\Rightarrow$ local model for Hitchin space

$$\mathcal{M}_H = T^*(G/\mathbb{T}) \qquad \mathbb{T} \subset G \text{ max torus}$$

$$= \tilde{N} \quad \text{Springer resolution}$$

$$= \mathcal{O}^{reg} \quad \text{regular semisimple orbit in complex sp.}$$

$$[\omega_I] = \alpha \qquad [\omega_J] = \beta \qquad [\omega_K] = \gamma$$

$\alpha, \beta, \gamma \in \mathfrak{t}$ Lie algebra of torus

$(H^2(\mathcal{M}_H) \simeq \mathfrak{t})$

What's the Hilbert space $\mathcal{H}$ from this σ-model POV?

$\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}') =$ space of open string states between the two branes $\mathcal{B}_{cc}, \mathcal{B}'$ on $\mathcal{M}_H = T^*(G/T)$

Branes: $\mathcal{B}' =$ brane supported on $G/\mathbb{T} \subset \mathcal{M}_H$ zero section

$\mathcal{B}_{cc} =$ the canonical coisotropic brane

- Focus on A-model for $\omega = \omega_K$

Topological A-branes: two types

- Lagrangian A-branes (e.g. $\mathcal{B}'$)
- (Kapustin-Orlov) A-branes with curvature F of Chan-Paton bundle s.t.

$$(F\omega^{-1})^2 = -\mathbb{1} \quad \underline{\text{coisotropic}}.$$

$\mathcal{B}_{cc}$: supported on $\mathcal{M}_H = T^*(G/T)$, $F = \omega_J$

A-model on $\mathcal{M}_H = T^*(G/T)$, ω_K

$\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$. In this example

$\mathcal{B}_{cc}$ & $\mathcal{B}'$ are of type (A,B,A)

$\rightsquigarrow$ $\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$ = holomorphic sections of a line bundle on G/T.

$\{A_{\mathbb{C}}\text{-branes}\} \longrightarrow \{\text{reps of } G_{\mathbb{R}} \subset G_{\mathbb{C}}\}$

$\mathcal{B} \longmapsto \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B})$

$\left\{\begin{matrix} K_{\mathbb{R}}\text{-invariant A-branes} \\ \text{on } \mathcal{M}_{H,\lambda} \end{matrix}\right\} \longrightarrow \left\{\text{HC modules}\right\}_{\lambda}$

inf. character

$\lambda = \eta + i\gamma$

Example: $G_{\mathbb{R}} = SL_2\mathbb{R}$

A-branes on $\mathcal{M}_H = \mathbb{H}^2 /\!/\!/ U(1) = T^*S^2$

invariant under $K_{\mathbb{R}} = SO_2$

"Gibbons-Hawking" form

$$ds^2 = H(d\vec{x})^2 + H^{-1}(d\chi + \vec{a})^2$$

$\vec{x} \in \mathbb{R}^3$ $\chi \in [0, 2\pi]$

$$H = \frac{1}{2|\vec{x} - \vec{x}_0|} + \frac{1}{2|\vec{x} + \vec{x}_0|} \qquad \vec{\nabla} \times \vec{a} = \vec{\nabla} H$$

$\vec{x}_0 = (\alpha, \beta, \gamma) \in \mathbb{R}^3$

$\vec{\omega} = (\omega_I, \omega_J, \omega_K) = (d\chi + \vec{a}) \cdot d\vec{x} - \frac{1}{2} H d\vec{x} \times d\vec{x}$

$K_{\mathbb{R}}$-invariant A-branes:

invariance $\Rightarrow$ represented by line $\ell \subset \mathbb{R}^3$ times the circle $K_{\mathbb{R}}$

To be A-branes: take $\ell \subset \mathbb{R}^3$ a planar curve (living in plane $x_3 = \text{const}$)

--- drawn in symplectic quotient $T^*S^2 /\!/ K_{\mathbb{R}}$

Depending on λ may or may not have special points in this plane:

get only cylinder A-brane $\longleftrightarrow$ principal series. If have special points have branes ending at these points (Lefschetz thimbles) $\rightsquigarrow$ discrete series or brane interpolating two of them:

$\Longleftrightarrow$ compact brane, fin dim representation.