A. Kapustin - Holomorphic-Topological Twisting of $N=2$ gauge theories

Note Title 11/28/2007

History: want to generalize electric-magnetic duality of Maxwell's equations to nonabelian gauge theories: S-duality

Examples of such conjectures:

1. Montonen-Olive duality of $N=4$ SYM
   - maximally supersymmetric, 4 copies of spinor generators Q_α^i $i=1,\dots 4$ (spinors of Spin 4) — such theories completely determined by
     - choice of gauge group G
     - gauge coupling $g^2 \leadsto \hbar$
     - θ angle

   M-O duality: theory for gauge group G is isomorphic to that for ${}^LG$ Langlands dual.

2. S-duality for $N=2$ SYM

$N=2$: two supercharges Q^i_α $i=1,2$
Many more such theories than $N=4$:
- Gauge group G
- R fin dim rep. of G:

can have different multiplets under SUSY algebra: gauge field $\in$ adjoint, other fields can be in a rep. R

Bosonic fields:
- A connection on E G-bundle
  $\phi \in \operatorname{ad} E_{\mathbb{C}}$ complex adjoint field (Higgs)
- Matter $q \in R(E)$, $\tilde{q} \in R^\vee(E)$
  associated bundles in representation R & dual.
  – have also a mass parameter.

Recover $N=4$ when $R=$ adjoint & mass $=0$.

Do these theories have S-duality?
Obstacle: coupling can run under renormalization. So we'll restrict to those where gauge coupling is energy independent – <u>finite</u> $N=2$ theories

What condition does finiteness impose on R?
Need index of R (action of Casimir)
to be same as that of adjoint.
Only finitely many such for given G.
Don't know if these should have S-duality
in general, but do expect for many..

e.g. $G = SU_2$ $R = V_2 \otimes \mathbb{C}^4$:
Seiberg-Witten theory.

S-duality: the theory is self-dual,
nontrivial action on states of the theory

These theories do have topological twist ---
Donaldson-Witten twist ..... doesn't appear related
to S-duality. No analogue of GL topological
twist!

But we only really want (for Langlands type
applications) to twist not on all 4-manifolds
but on products $M_4 = \Sigma \times C$ — more
options.

To twist need homomorphism

$\rho: U(1)_C \times U(1)_\Sigma \longrightarrow$ symmetry group

- get a theory which has fermionic symmetry only on such product manifolds. (easier than looking for $SO(4) \to$ symmetry !)

Symmetries: SU_2 acts rotating Q_i, & have another $U(1)$ R-symmetry when theory is finite (call it $\widetilde{U(1)}_R$)

- identify $U(1)_C \simeq U(1)_R \subset SU(2)_R$

& $U(1)_\Sigma \simeq \widetilde{U(1)}_R$

After this modification: A remains a connection,

ϕ lives in $\mathrm{ad}\, E_C \otimes K_\Sigma$

q lives in $R(E) \otimes K_C$

$\tilde{q}$ lives still in $R^\vee(E)$.

... specializes to GL twist when $R =$ adjoint

($\tilde{q}$ becomes adjoint scalar σ in $N=4$ GL twist)

$Q_{BRST} = u\,Q_\ell + v\,Q_r \qquad u, v \in \mathbb{C} \quad$ (up to scale)

Special points: $t = \frac{v}{u} = 0$ or ∞.

Action: $S = \underbrace{\{Q, -\}}_{\substack{Q\text{-exact}\\ \text{piece}}}$ + piece independent of Kähler form ω_Σ, ω_C. & complex structure on Σ

$\Rightarrow$ so theory is volume independent, but BRST transformations depend on complex structure on C.

TFT: Suppose $\mathcal{O}^{(0)}$ (0-form) is a BRST-closed observable $\{Q, \mathcal{O}^{(0)}\} = 0$

$\Rightarrow$ $d\mathcal{O}^{(0)}$ is Q-exact (Witten) $\Rightarrow$

get nonlocal observables by integrating over cycles: the variation $\delta \int_\gamma \mathcal{O}^{(1)} = 0$ where $d\mathcal{O}^{(0)} = \{Q, \mathcal{O}^{(1)}\}$.

In Σ direction our theory is topological $\Rightarrow$

$d_\Sigma\, \mathcal{O}^{(0)} = \{Q, \mathcal{O}_\Sigma^{(1)}\}$ observables, so for cycle $\gamma = \gamma_\Sigma \times p_C$ get nonlocal operator $\int_\gamma \mathcal{O}^{(1)}$. On C side can use $\bar\partial$ instead:

$$\bar\partial_C \mathcal{O}^{(0)} = \{Q, \mathcal{O}_C^{(1)}\}, \quad \mathcal{O}_C^{(1)} \in \Omega^{0,1}_C$$

Let $\beta \in H^0(\Omega^{1,0}_C)$, $\delta \int_C \beta\, \mathcal{O}_C^{(1)} = 0$

well defined functional ... ie observable taking values in $H^1(\mathcal{O}_\Sigma)$.

Correlators will now be <u>holomorphic</u> functions of insertion points on C.

At special points $t=0$ & $t=\infty$ theory is <u>just</u> holomorphic, not holomorphic-topological ... ie not topological in Σ either ... looks like 4d generalization of chiral algebra.

<u>Reduction to 2d</u>: $\Sigma \times C$, can take volume of either to zero. [if generic]

1. vol $\Sigma \to 0$: get effective field theory on C

- holomorphic field theory in one complex dimension.
ie holomorphic part of a CFT $\Longleftrightarrow$ chiral algebra. In classical approximation: cohomology of chiral differential operators on the Hitchin moduli space $\mathcal{M}_H(G, \Sigma)$ equipped with vector bundle $\mathcal{E}_R$ depending on R.
(fermions take value in $\mathcal{E}_R$)
Malikov-Schechtman-Gorbounov: need
$p_1(\mathcal{E}_R) = p_1(T_{\mathcal{M}_{Hit}}(G, \Sigma))$: follows from anomaly cancellation condition on R.

2 Vol $C \longrightarrow 0$: get an effective TFT on Σ, namely B-model with target $\mathcal{M}_{Higgs}(G, R, C)$
- reduces to Hitchin space when R = adjoint.

Anomaly cancellation $\Longrightarrow$ this is a Calabi-Yau moduli space.
Generalized Higgs bundles: $\varphi \in R(E) \otimes K_C$

- $F - (q \otimes q^{\dagger})_{ad} = 0$
- $\bar{\partial}_A q = 0$

modification of vortex equation.

Can take invariant polynomials in q $\Rightarrow$ map to some affine variety.

Hitchin: moduli of stable generalized Higgs is same as solutions to above equations.

<u>S-duality</u> The TFT on Σ admits B-branes:

- the derived category of Coh $\mathcal{M}_{Hit}(G,R,C)$

So when (G,R) theory is S-dual to $(\tilde{G},\tilde{R})$ theory should get

$$D^b(\mathcal{M}_{Higgs}(G,R,C)) \simeq D^b(\mathcal{M}_{Higgs}(\tilde{G},\tilde{R},C))$$

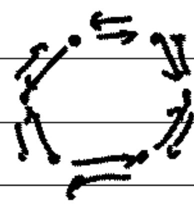
<u>Examples</u> • $R =$ adjoint : $\tilde{G} = {}^LG$, $\tilde{R} =$ adjoint

$\Rightarrow$ classical limit of geometric Langlands (Donagi–Pantev)

• Quiver theories: $G = \underbrace{SU(N)\times\cdots\times SU(N)}_{k \text{ times}}$

$R = \bigoplus_{i=1}^{k} V_i \otimes V_{i+1}^*$

$V_{k+1} = V_1$



exactly affine ADE quivers give finite $N=2$ quiver theories [Katz-Klemm-Vafa]

Duality group (autoequivalences of the theory) is the mapping class group of $\mathcal{M}_{1,k}$ elliptic curve with k marked points (elliptic braid group) — should act by autoequivalences!

- actually generic element of the duality group changes the Lie group (with fixed Lie algebra).

e.g. could take instead $\tilde{G} = \dfrac{SU(N)\times\cdots\times SU(N)}{\mathbb{Z}_N}$

should be derived equivalent with above!

Line analogs of 't Hooft - Wilson operators:

no BRST invariant Wilson lines on C

but some on Σ:

pick $p_c \in C$ γ closed loop on Y, ρ rep of G

$$\Rightarrow \quad W_{\rho, p_c, \gamma} = \mathrm{Tr}_\rho \, \mathrm{Hol}(A + i\varphi) \quad \text{holonomy}$$

These act by functors on branes.

S-duality maps these to Hecke transformations

(preserve regularity of Higgs fields).

- interesting already for R = adjoint.