E. Witten - Geometric Langlands & TFT

Note Title 11/28/2007

[Most naturally: start from 6 dimensions
or at least 4 —
today: stick to 2 dimensions!]

Goal: Wild ramification

G = complex reductive Lie group
C = oriented 2-manifold, compact, closed
$\mathcal{Y}(C,G) = \{ \mathrm{Hom}(\pi_1(C), G) \}$ mod conjugation
[really stable ones]

(Baulch's talk) $\mathcal{Y}$ is a complex symplectic manifold

Complex since G complex, symplectic from H^1:

$T_E \mathcal{Y} = H^1(C, \mathrm{ad}\, E)$ for $E \to C$ flat bundle

$$\underset{\in}{\alpha, \beta} \longmapsto \Omega(\alpha,\beta) = \int_C \mathrm{Tr}\, \alpha \wedge \beta$$

where Tr is an invariant bilinear form on $\mathfrak{g} = \mathrm{Lie}(G)$. The cohomology class of Ω is real .... so $\mathrm{Re}\,\Omega$ is nontrivial in cohomology but $[\mathrm{Im}\,\Omega] = 0$

2d TFTs:

a. B-model on Y: need $c_1(Y)=0$
but in fact Y is even complex symplectic
(B-models makes sense for some noncompact varieties) .... no choices

b. A-model on Y: pick the real symplectic structure $\omega = \mathrm{Im}\,\Omega$
.... superficially depends on choice of Tr
but it doesn't really: naively since $[\mathrm{Im}\,\Omega]=0$. (need to be careful at ∞)
– but actually (Y,ω) independent of Tr up to a symplectomorphism unique up to homotopy.

More generally we could define a 1-complex-param. family of models with complexified Kähler class (ie add B-field)

$$\omega_{\mathbb{C}} = \mathrm{Im}\,\Omega + \Psi\,\mathrm{Re}\,\Omega \qquad \Psi \in \mathbb{C}$$

(Kapustin–Witten) $\rightsquigarrow$ quantum geometric Langlands
.... need to choose Tr to identify Ψ with a number

(really the form is $\Psi \cdot \mathrm{Tr}$)

As $\Psi \longrightarrow \infty$ go to B-model !

With hindsight (& following Hausel-Thaddeus) consider:

instead of G take ${}^L G$,

is the B-model of G mirror to A-model of ${}^L G$

& vice versa?

– to answer it need more structure! pick

a complex structure on C

[Note we're considering a 2d TFT:

maps $\Sigma \longrightarrow Y = Y(C)$ so two surfaces

involved : Σ - family of flat connections on

C best understood via fields on $\Sigma \times C$]

Hausel Thaddeus (following Harvey Moore Strominger,

Vafa Bershadsky Johansen) :

use Hitchin's equations –

Topologically : flat G-bundle $E \to C$

Reduce structure group of E from G to G_{cpt}

– no problem in doing so topologically since G/G_{cpt} contractible.

But Hitchin's eqn give canonical reduction given $\mathbb{C}$-structure on C!

$\exists!$ "harmonic" reduction $G \to G_{cpt}$

(harmonic section of associated G/G_{cpt} bundle)

If $\mathcal{A}$ = flat connection, then the reduction amounts to writing $\mathcal{A} = A + \phi$

where A is a G_{cpt} connection &

$\phi \in \Omega^1_{\mathfrak{g}_{cpt}}$ [real & imaginary parts of $\mathcal{A}$]

Hitchin's equations: $\begin{cases} F - \phi \wedge \phi = 0 \\ d_A \phi = 0 \\ d_A * \phi = 0 \end{cases}$

Write $\phi = \phi^{1,0} + \phi^{0,1}$. Find $\bar{\partial}_A \phi^{1,0} = 0$

holomorphic $\rightsquigarrow$ Higgs bundle

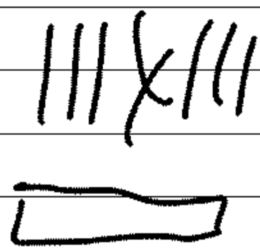
(E, φ): (hol. G-bundle, $\varphi \in H^0(C, \mathrm{ad}\, E \otimes K_C)$)

$\rightsquigarrow$ Hitchin's fibration

$(E, \varphi) \longmapsto$ char. poly of $\varphi \in B$ Hitchin base

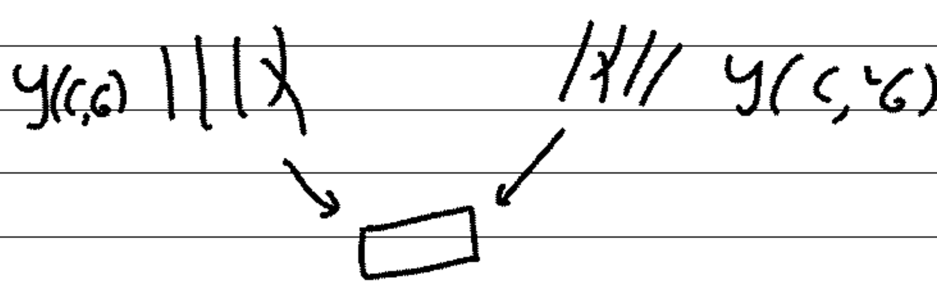
(B is a linear space)

$$\begin{array}{c} \mathcal{Y} \\ \downarrow \\ B \end{array}$$



completely integrable system
- fibers are special Lagrangian submanifolds
- in $\mathcal{Y}$ complex structure they're not holomorphic but are SLAGs for $\omega = \mathrm{Im}\, \Omega$.

So we're in the Strominger-Yau-Zaslow picture for mirror symmetry! The bases for the Hitchin fibrations for G, ${}^LG$ are same



Fibers are dual tori (Hausel-Thaddeus for SL_n, Donagi-Pantev in general)

This mirror symmetry is in fact <u>independent</u> of the complex structure on C.

[Deligne question: does the geometry of the singularities of Hitchin fibers depend sensitively on the complex structure on C?]

Tame ramification: $C' = C \setminus p$

[Gukov-Witten] $\mathcal{Y}(C', G; \mathcal{U}) = \{ \mathrm{Hom}(\pi_1(C'), G)$ s.t. monodromy around p is in the conjugacy class $\mathcal{U} \subset G \} / \sim$

Again have Ω complex symplectic, $\mathrm{Im}\,\Omega$ cohomologically trivial
$\Rightarrow$ get A-model ($\omega = \mathrm{Im}\,\Omega$)
& B-model

Q: Is there some correspondence

$U \in G \longleftrightarrow {}^L U \subset {}^L G$

st B-model of $\mathcal{Y}(C', G; U) \longleftrightarrow$

A-model $\mathcal{Y}(C'; {}^L G; {}^L U)$?

No: • no correspondence between conjugacy classes

• B-model holomorphic in U but A-model not holomorphic in ${}^L U$.

Need Hitchin's equations with a singularity at p (coord $z=0$ $z = re^{i\theta}$) — studied by Simpson.

$\alpha, \beta, \gamma \in \mathfrak{t}_{cpt}$ torus of compact group

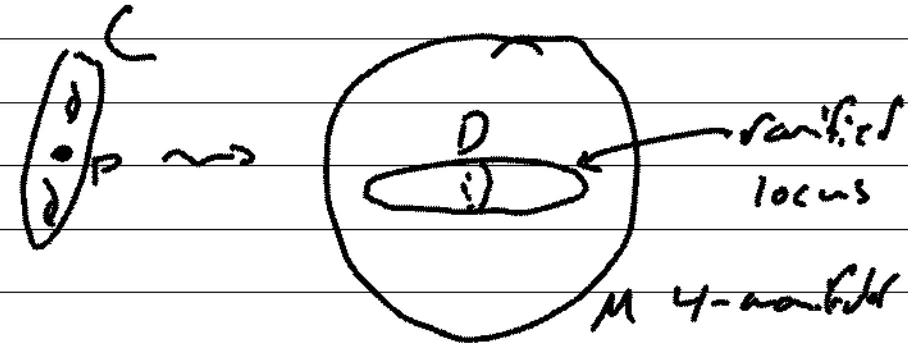
Write $A = \alpha \, d\theta + \dots$

$\phi = \beta \frac{dr}{r} + \gamma \, d\theta + \dots$

$|A|, |\phi| \sim \frac{1}{|z|}$.

Monodromy $u = \exp(-2\pi(\alpha - i\gamma))$

4d picture



- codim 2 singularity (surface operator) in 4d gauge theory, $\rightsquigarrow$

G_{cpt} bundle on M, reduces to T_{cpt} on D

$\rightsquigarrow$ T_{cpt} bundle $V \longrightarrow D$

$c_1(V)$ lives in lattice. $\eta : \Lambda \longrightarrow U(1)$

$\Longleftrightarrow$ $\eta \in {}^L T_{cpt}$

can define $\exp(i(\eta, c_1(V))$ $\cdots$ something (SUSY) to be included in path integral (θ-angle)

[$\eta \longleftrightarrow$ flat $U(1)$ gerbe over stack of bundles]

Good set of parameters: $(\alpha, \beta, \gamma, \eta)$

Can now follow Hitchin/mirror story as before.

Wild ramification

E flat G-bundle over $C' = C \setminus p$ with monodromy
- can extend over C holomorphically
- ie have gauge where $A^{(0,1)}$ is smooth at p

but $A^{(1,0)} = (\alpha - i\gamma)\frac{dz}{z} + \dots$
simple pole.

Can now consider instead irregular singularities:
$E \to C$ hol. G-bundle with a hol connection

$$D = \frac{\partial}{\partial z} + \frac{T_n}{z^n} + \frac{T_{n-1}}{z^{n-1}} + \dots + \frac{T_1}{z} + (\text{reg.})$$

pole of order n

Want to define a moduli space of such things Y such that Y is a complex symplectic manifold (locally) independent of the complex structure of C .... Boalch, Biquard-Boalch: have a version of Hitchin's equations in this setting!

How can this be right? What other topological invariants does a connection have beyond monodromy? $\rightsquigarrow$ Stokes matrices

$\rightsquigarrow$ get invariants in $U_\pm$ & T

upper-lower triangular matrices & torus.

Mirror symmetry is independent locally of the parameters

$\rightsquigarrow$ get action of braid group of G in addition to mapping class group.