Tom Nevins - Perverse Bundles & CM Spaces

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Hilbert schemes:

$$(\mathbb{C}^2)^{[n]} = \{ I \subset \mathbb{C}[x,y] : \dim_{\mathbb{C}}(\mathbb{C}[x,y]/I) = n \}$$
$$\subset Gr(\mathbb{C}[x,y], n).$$

Reminder: to study configs of n points in $\mathbb{C}^2$: start w/ $(\mathbb{C}^2)^n/S_n$ - but that's singular:

better to be clever: study ideal of functions vanishing at collection of points.

$\longrightarrow$ codim n ideals.

Extra structure beyond locations of points

e.g. $2\cdot\{0\}$: $(x,y)^2 \subseteq I \subseteq \mathbb{C}[x,y]$ (codim 3)

$\rightarrow$ pick any line in Span $\{x,y\}$ gives an ideal - ie get point & cotangent direction: $(x,y)/(x,y)^2 = T_0^*\mathbb{C}^2$.

Elementary description:

$I \longleftrightarrow$ quotient $\quad \mathbb{C}[x,y] \xrightarrow{\pi} \mathbb{C}[x,y]/I \cong \mathbb{C}^n$

Module structure: two $n \times n$ matrices

$X, Y \in \mathfrak{gl}_n \quad , \quad [X,Y]=0$

$i = \text{image}(1) \in \mathbb{C}^n$

Surjectivity of $\pi \iff i$ is cyclic

$$(\mathbb{C}^2)^{[n]} = \left\{ X,Y \in \mathfrak{gl}_n ,\ [X,Y]=0 \ \& \ i \in \mathbb{C}^n \atop (s) \quad i \text{ cyclic vector} \right\} \Big/ GL\,\mathbb{C}$$

More symmetric description:

$$M = \{ (X,Y,i,j) : X,Y \in \mathfrak{gl}_n\,\mathbb{C}, \quad i \in \mathbb{C}^n, j \in (\mathbb{C}^n)^* \}$$
$$= \mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathbb{C}^n \times \mathbb{C}^n$$

$\downarrow \mu \qquad \downarrow$

$\mathfrak{gl}_n \qquad [X,Y] + ij$

So $(\mathbb{C}^2)^{[n]} \subset \mu^{-1}(0) / GL_n$

in fact $(\mathbb{C}^2)^{[n]} = \mu^{-1}(0)^s / GL_n$:

impose condition (s),

which <u>implies</u> $j = 0$

Note : $M = T^*(\mathfrak{gl}_n \times \mathbb{C}^n)$

$GL_n \curvearrowright \mathfrak{gl}_n \times \mathbb{C}^n$ & action is induced action on T^* & μ is moment map.

Q: What if we relax stability condition (s)?

Orbit closure of $p \in \mu^{-1}(0)^s$ will contain something with $i = 0$

So in $\operatorname{Spec} \mathbb{C}[\mu^{-1}(0)]^{GL_n}$ Hilbert

scheme locus gets identified with orbits of GL_n on $(X, Y, 0, 0)$ with $[X, Y] = 0$

– that's just $(\mathbb{C}^2)^n / S_n$.

So $(\mathbb{C}^2)^{[n]}$ gets contracted to $S^n(\mathbb{C}^2)$.

⇝ not so interesting

Better thing to do: take stack quotient

Construction build complex

$C: \mathbb{C}[x,y]^n \xrightarrow{\phi} \mathbb{C}[x,y]^n \oplus \mathbb{C}[x,y]^n \oplus \mathbb{C}[x,y] \xrightarrow{\psi} \mathbb{C}[x,y]^n$

$\phi = \begin{pmatrix} x - X \\ y - Y \\ j \end{pmatrix}$ $\psi = (y - Y \quad X - x \quad i)$

get complex $\iff$ $[X,Y] + ij = 0$.

$j = 0 \iff$ can reconstruct our module:

this complex C is quasiisomorphic to

$$[\, \mathbb{C}[x,y] \longrightarrow \mathbb{C}^n \,] \quad \text{if } j = 0$$
$$f(x,y) \longmapsto f(X,Y);$$

So if $j=0$ & i cyclic

$H^0(C) =$ our ideal I, & $H^1(C) = 0$

In general get some complex C such that

1. $H^0(C)$ is torsion-free (rank 1)
2. $H^1(C)$ is 0-dimensional in support.
3. $H^i(C) = 0$ for $i \neq 0, 1$.

<u>Perverse line bundle</u> on $\mathbb{C}^2$:

complex of modules over $\mathbb{C}[x,y]$

e.g. $\mathbb{C}[x,y] \xrightarrow{\lambda} \mathbb{C}$ as $\lambda \to 0$ becomes such!

[Here n is the second Chern class of C]

Facts: a. $\mu^{-1}(0)/GL_n$ is the moduli stack of perverse line bundles with $c_2 = n$ on $\mathbb{C}^2$.

b. $\mu^{-1}(0)/GL_n$ is the cotangent bundle of $\mathfrak{gl}_n \times \mathbb{C}^n / GL_n$.

... in general if $G \circlearrowright M$

$G \circlearrowright T^*M$ induced action

$\Rightarrow$ $T^*M \xrightarrow[\mu]{} \mathfrak{g}^*$ moment map

& $T^*(M/G) = \mu^{-1}(0)/G$

provided M/G reasonable.

Q2 What if we replace $\mu^{-1}(0)/G$ by $\mu^{-1}(\xi)/G$? $\xi = \hbar\, \mathrm{Id}$

e.g. What is $\mu^{-1}(\mathrm{Id})/GL_n = \{X, Y, i, j : [X,Y] + ij = I\}/GL_n$

Approximate: $[X,Y] = Id$

$\rightsquigarrow$ $\mathbb{C}\langle X,Y\rangle / [X,Y]=1 \longrightarrow \mathfrak{gl}_n \mathbb{C}$

$\quad\quad \mathcal{D}(\mathbb{A}^1) = \mathbb{C}[x, \frac{\partial}{\partial x}]$ Weyl algebra.

— there are none such!

no solutions for $i=0$.

<u>Amazing fact</u> Actually $\mu^{-1}(Id)/GL_n$ is the moduli space for (right) ideals of $\mathcal{D}(\mathbb{A}^1)$ of $c_2 = n$.

equivalently could say perverse line bundles on $\mathcal{D}(\mathbb{A}^1)$

What kind of gadget is this?

$\mu^{-1}(I)/GL_n$ ⟲ $\mu^{-1}(0)/GL_n$ affine space

$\searrow \quad \swarrow$

$\mathfrak{gl}_n \times \mathbb{C}^n / GL_n$

pseudo principal bundle for $T^*(\mathfrak{gl}_n \times \mathbb{C}^n / GL_n)$

Why study these?
cones out of integrable systems
- rational solutions of KP equation
$u = u(x,y,t)$
$$(u_t - uu_x - 6u_{xxx})_x = u_{yy}$$

Rational solutions can be encoded in
$D(\mathbb{A}^1)$-modules (Sato, BZ-N)
We described moduli space of such things
& more general $D(C)$-modules for
C a genus 1 curve possibly singular
($\mathbb{A}^1 \subset C$ cuspidal curve)
via a Fourier transform, & showed
that it's same as phase space of
a particle system

Amazing fact first proven by Berest-Wilson
matrix description identifies with phase space
of particle system.

Theorem [Bz·N] All of the above, directly, replacing $\mathbb{A}^1$ with any smooth curve.

$\mathfrak{gl}_n = \mathbb{C}^n / Gl_n$: n-dim module over $\mathbb{C}[x]$ (length n torsion sheaf) on $\mathbb{A}^1$ + section.

Moduli of perverse line bundles on $T^*\mathbb{A}^1$ $= T^*\left(\begin{Bmatrix}\text{length } n \text{ torsion} \\ \text{sheaves + section}\end{Bmatrix}\right)$

moduli of $\mathcal{D}_{\mathbb{A}^1}$-mod = twisted cotangent of length n torsion sheaves + section

$\mathcal{D}_{\mathbb{A}^1} = \mathcal{O}(T^*_\hbar \mathbb{A}^1)$

<u>General</u> Same for any curve C (& higher rank)

Method : Beilinson transform : duality along fibers of $T^*_\hbar C \longrightarrow C$, $\mathbb{P}^1$ bundle.