G topological group ... e.g. $SL_2\mathbb{R}$, $SL_2\mathbb{F}_p$, $SL_2\mathbb{Q}_p$

A *representation* of G is a complex vector space V, ~~with~~ in fact a topological vector space/$\mathbb{C}$

& $\pi : G \longrightarrow \mathrm{Aut}\, V$ continuous linear automorphisms
s.t. $G \times V \longrightarrow V$ is continuous

- Finite dimensional representations: V fin dim has a unique topology ---> just looking at continuous maps $G \longrightarrow GL(V) \cong GL_n\mathbb{C}$. (homomorphisms) - realize G by matrices

- Map of representations $\varphi : V \longrightarrow W$: continuous map compatible with G action, i.e. $g \cdot \varphi(v) = \varphi(g \cdot v)$.

- Subrepresentation: closed linear subspace $W \subset V$ preserved by G.

Irreducible: no nontrivial subrepresentations.

Indecomposable: V is not $W_1 \oplus W_2$ direct sum of subrepresentations.

This is weaker notion: can have $W \subset V$ with no invariant complement.

Example: $G = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = U$ "unipotent" $\subset GL_2\mathbb{C}$

$V = \mathbb{C}^2 \supset W = \begin{pmatrix} * \\ 0 \end{pmatrix}$ preserved by U, no complement.

trivial rep $W \hookrightarrow V \twoheadrightarrow V/W$ trivial rep

[trivial : every $g \in G$ acts by Id]

- nontrivial extensions.

Unitary representation: V is a Hilbert space & G acts by unitary transformations, $\langle gv, gw\rangle = \langle v, w\rangle$.

Unitary reps: irreducible $\longleftrightarrow$ indecomposable:

Given $W \subset V$ can take $W^\perp \subset V$, $V = W \oplus W^\perp$ sum of representations

Topological vector space: Hilbert, Banach, Fréchet: want Hausdorff, $\boxed{\text{complete, locally convex}}$: topology is induced by a family of seminorms $\rho : V \to \mathbb{R}$ $\quad \rho(x+y) \le \rho(x) + \rho(y)$, $\rho(\lambda x) = |\lambda| \rho(x)$

Reason: can integrate $f : [a,b] \to V \Rightarrow \int_a^b f \in V$

(* ~~Idea~~ G finite, or more generally compact: reps are completely reducible: any V is $\oplus V_i$ of irreps, and all irreps are unitary & finite dimensional)

G abelian: irreps fin dim $\qquad$ G compact: irreps discretely

Origin of reps: X topological space, $\quad$ Riemannian?

$G \curvearrowright X$ continuous $\Rightarrow$

$G \curvearrowright \mathrm{Fun}(X)$ acts on spaces of functions!

$(g \cdot f)(x) = f(g^{-1}x)$

$$(h \cdot (g \cdot f))(x) = f(h^{-1} g^{-1} x)) = f((gh)^{-1} x) \overset{\shortparallel}{} (gh) \cdot f(x)$$

Examples: $C_c(X)$, $C_c^\infty(X)$ locally convex

$L^2(X)$ wrt μ measure – Hilbert $\qquad$ μ G-invariant

$L^p(X)$ $p \ge 1$ Banach

$C^{-\infty}(X)$ distributions, $C^\omega(X)$ real analytic, holomorphic functions, sections of (equivariant) line bundles ... algebraic

– much common structure to all of these, break into two parts: • algebra • topology (Harish-Chandra)

Problems of "Representation Theory":

Representation Theory 1. Describe irreducible unitary reps of G, all irreps, all indecomposables, full structure of extensions & maps between representations

- Geometric realization: construct reps as functions on some space, sections of bundles etc.. Borel-Weil etc.
- Character theory: f.d. rep is determined by $\mathbb{P}^1, \mathbb{H}$ ...

a function $\chi_V(g) := tr_V(g)$, conjugation invariant. Describe character concretely, e.g. restriction to diagonal matrices (GL_n) or in basis of class functions given by conjugacy classes

H-C: character theory for ∞-dim reps

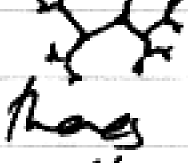
- classes of reps: unitary $\subset$ admissible, latter controlled by algebraic data + topology: infinitesimal equivalence $(\mathfrak{g}, K)$-modules, then model in different function spaces.

2. Harmonic analysis: Given a particular G-space X, decompose $Fun(X)$ as a G-module, e.g. write $L^2(X)$ in terms of irreducible unitary representations

$L^2(G)$: Plancherel formula, Nonabelian Fourier theory

$SL_2\mathbb{Z} \backslash SL_2\mathbb{R}$, $\Gamma \backslash G$ spectral decomposition: Hecke, Weil, Tate... Langlands: number theory via harmonic analysis.. $SL_2\mathbb{Q}_p$ all p & $SL_2\mathbb{R}$ theory fit together to Langlands program.

Modular forms (Eisenstein, cusp, ...) are special representations of $SL_2\mathbb{R}$ in here..

$\mathbb{H}$ vs 

- universal theory in rep theory, similarity across fields -- again H-C, Hecke algebra...

Why SL_2? ... like asking Why Riemann Surfaces..

- First simple group, see many phenomena here concretely
- all semisimple groups & their reps "built out of SL_2's
- modular forms best known part of Langlands
- hyperbolic geometry & Riemann surfaces
- Bessel, hypergeometric functions etc - many special fns come from SL_2 as Eigenfunctions of differential operators ....

- Special roles: Lefschetz SL_2 action
- oscillator / metaplectic rep:
  $\frac{1}{2}\Delta, \quad r\frac{\partial}{\partial r} + \frac{n}{2}, \quad \frac{1}{2}r^2$ on $\mathbb{C}[x_1, \dots x_n]$
  give action of $\mathfrak{sl}_2\mathbb{R}$ or $\widetilde{SL_2\mathbb{R}}$.
  - applications to classical harmonic analysis!
  wave eqn / Huygens' principle, etc...
- $PSL_2\mathbb{C} = SO(1,3)^+$ Lorentz group
  - role in quantum field theory: particles = irreps of $PSL_2\mathbb{C} \ltimes \mathbb{R}^4$.

Lie algebras

Example $\mathbb{R}^3$, $\times$. skew symmetric,
not associative: $(i\times j)\times j = -i$ $\quad i\times(j\times j)=0$
Satisfies Jacobi identity.

Proof!
$(a\times b)\times c = (a\cdot c)b-(b\cdot c)a$

$$[[a,b],c] + [[b,c],a] + [[c,a],b]=0$$

or better $\quad [a,[b,c]] = [[a,b],c] + [b,[a,c]]$

Call $[a,-] = \partial_a$, $\quad [\ ] : \cdot\ \cdot$
so $\quad \partial_a(b\cdot c) = (\partial_a b)\cdot c + b(\partial_a c)$

Leibniz rule: $\quad \partial(fg) = (\partial f)g + f(\partial g)$.

~~Given commutative product ... define~~

Given a manifold M, eg $\mathbb{R}^2$, look at vector fields on M: in local coordinates, $\sum f_i \frac{\partial}{\partial x_i}$.

Act on fns by differentiation. Can't compose:
$\frac{\partial}{\partial x}\frac{\partial}{\partial y} f$ is not first order! but can take

Commutator: $\xi,\eta \longmapsto [\xi,\eta] = \xi\eta - \eta\xi$.

$[\partial_x,\partial_y]=0$

$$[y\partial_x, \partial_y] = y\partial_x\partial_y - (\partial_y(y\partial_x))$$
$$= y\partial\partial - (y\partial_y\partial_x + \partial_x) = -\partial_x.$$

Mixed partials agree $\Rightarrow$ leading terms always cancel, left with first order.

Def: Derivation of a product $\bullet$ — Leibniz.

Derivations have commutators which are derivations. & commutator itself satisfies Leibniz rule!
also Jacobi

Example Riemann sphere $f(z)\partial/\partial z$ holomorphic vector fields

$$w = \frac{1}{z} \quad \frac{\partial}{\partial w} = \frac{dz}{dw}\frac{\partial}{\partial z} = -\frac{1}{w^2}\frac{\partial}{\partial z} = -z^2\frac{\partial}{\partial z}$$

So $f(z)\,\partial_z$ f polynomial degree $n \Rightarrow$ pole order n at ∞
$\Rightarrow$ need $n \le 2$

$$\Rightarrow \quad e = \frac{\partial}{\partial z}, \quad h = -2z\frac{\partial}{\partial z} \quad f = -z^2\frac{\partial}{\partial z}$$

$$[e,f] = -2z\frac{\partial}{\partial z} = h$$

$$[h,e] = 2e \qquad [h,f] = 4z^2\frac{\partial}{\partial z} - 2z^2\frac{\partial}{\partial z} = 2z^2\frac{\partial}{\partial z} = -2f$$

$$\boxed{\begin{aligned}[][e,f]&=h\\ [h,e]&=2e\\ [h,f]&=-2f\end{aligned}}$$

$$i \mapsto \tfrac{1}{\sqrt{2}}\, i h$$
$$j \mapsto \tfrac{1}{\sqrt{2}}(e-f)$$
$$k \mapsto \tfrac{1}{\sqrt{2}}(ie+if)$$

get isomorphism $(\mathbb{C}^3, \times) \cong \mathrm{Vect}(\mathbb{P}^1)$

Another source of Lie algebras: take matrices under $[AB] = AB - BA$. Clearly skew ... check Jacobi
– in fact for any associative algebra commutators form Lie $A \longrightarrow A^{Lie} = A, [,]$.

e.g. $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

basis for matrices with trace $= 0$.

check commutators!

Representation of a Lie algebra: $\underset{[,]}{\mathfrak{g}} \longrightarrow \underset{[,]}{\mathrm{End}\, V}$

e.g. $\mathrm{Vect}\,\mathbb{P}^1 \longrightarrow \mathrm{End}\,\mathbb{C}^2 = M_{2\times 2}$ 2-dim rep.
$\longrightarrow$ [illegible]

<u>adjoint rep</u> : $\mathfrak{g}$ acts on itself : Jacobi $\leftrightarrow$ action on itself is a homomorphism into matrices

$x \longmapsto [x, -]$

$[x,y] \longmapsto [[x,y],-] = [x,[y,-]] - [y,[x,-]]$

e.g. $\begin{bmatrix} e \\ h \\ f \end{bmatrix}$ basis for $\mathfrak{sl}_2$: $e \mapsto \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$h \mapsto \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix}$

$f \mapsto \begin{pmatrix} & & \\ -1 & & \\ & 2 & \end{pmatrix}$

How about $\mathbb{R}^3, \times$?

$\begin{pmatrix} i \\ j \\ k \end{pmatrix}$ $i \longmapsto \begin{pmatrix} & & \\ & & -1 \\ & 1 & \end{pmatrix}$ $j \mapsto \begin{pmatrix} & & 1 \\ & & \\ -1 & & \end{pmatrix}$

$k \mapsto \begin{pmatrix} & -1 & \\ 1 & & \\ & & \end{pmatrix}$ basis of <u>skew-symmetric matrices</u>

$\mathbb{R}^3, \times \cong \mathfrak{so}_3 \mathbb{R}$

$\mathfrak{gl}_n := M_{n \times n}$

$\mathfrak{sl}_n$: traceless

$\mathfrak{so}_n$: skew-symmetric $A = -A^t$

$\mathfrak{su}_n$: skew-hermitian $A = -\bar{A}^t$ (can't take $\mathbb{C}$-linear combos!!)

$\mathfrak{sp}_n$: $A \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} & I \\ -I & \end{pmatrix} A$ even # of dimensions

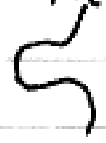
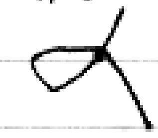
$\mathfrak{su}_2 \cong \mathfrak{so}_3$: $\mathfrak{su}_2$ $\begin{array}{ccc} i & j & k \\ \begin{pmatrix} i & \\ & -i \end{pmatrix} & \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} & \begin{pmatrix} & i \\ i & \end{pmatrix} \end{array}$

$\mathfrak{so}_3$ $\begin{array}{ccc} x & y & z \\ \begin{pmatrix} & & \\ & & -1 \\ & 1 & \end{pmatrix} & \begin{pmatrix} & & 1 \\ & & \\ -1 & & \end{pmatrix} & \begin{pmatrix} & -1 & \\ 1 & & \\ & & \end{pmatrix} \end{array}$

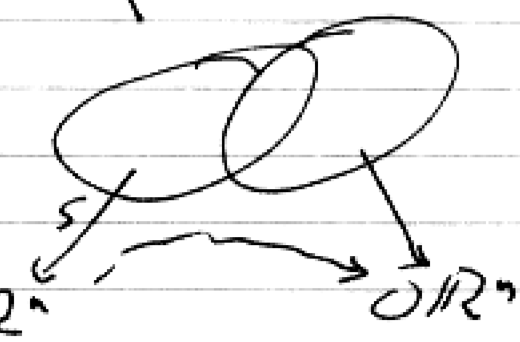
Lie Groups & Lie Algebras

Lie group: group of continuous symmetries in geometry, eg symmetries of Euclidean space = translations + rotations

More precisely: manifold with compatible group structure.
Manifold: locally looks like $\mathbb{R}^n$ for some n.

e.g  not 

Kinds of manifolds



how is gluing determined?

very interesting: complex manifold — $\mathbb{C}$, holomorphic maps
---> can talk about holomorphic fns.
$\Rightarrow$ complex Lie groups.

Matrix groups: closed subgroup of $GL_n\mathbb{R}$.

Actually every Lie group is locally a matrix group.

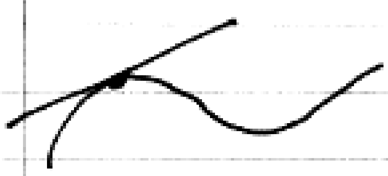
Locally $SU_2 \xrightarrow{2:1} SO_3$ SU_2 not realized as a ~~3dim~~ matrix group $\subset GL_3\mathbb{R}$.

$\widetilde{SL_2\mathbb{R}} \xrightarrow{\mathbb{Z}} SL_2\mathbb{R}$ not matrix group.

Compact Lie groups: e.g $U(1)$, $SU(2)$, $SO(3,\mathbb{R})$.

Connected: don't not usually G = finite...

Tangent space linear approximation to manifold



vector space all directions at a point

$\Longleftrightarrow$ path $\varphi : \mathbb{R} \to M$ $0 \mapsto x$

up to ones that agree to order two...

$f: M \to N$ smooth map $x \longmapsto Df: T_x M \to T_y N$

e.g. $a: G \times M \to M$ group acting

$\xi \in T_1 G$ Then for every $m \in M$ get $Df_m(\xi) \in T_m M$

$G \to M$

$g \mapsto g \cdot m$ vector field!

For example $M = G$ have right & left action
— two ways to get vector fields out of tangent vectors at origin...

Example $G = (S)O_n \subset Mat_{n \times n}$

$T_1 G \subset T_1 Mat_{n\times n} = Mat_{n \times n}$ is $\mathfrak{so}_n$ = skew-symmetric matrices!

Proof: A skew $\Rightarrow$ take $e^{tA} = 1 + tA + \frac{t^2}{2}A^2 + \dots$

$A + A^t = 0 \Rightarrow e^{tA} \cdot e^{tA^t} = 1$ orthogonal.

$\gamma : \mathbb{R} \to G$ $\gamma(0) = 1$, $\gamma'(0) = A$ get tangent vector

Conversely given such γ $\gamma^t \gamma = 1 \Rightarrow$

$$\gamma'(0)^t + \gamma'(0) = 0 \quad \text{skew.}$$

$T_g SO_n = g \cdot \text{Skew} = \text{Skew} \cdot g$.

Show $T_1 U_n$ = skew-hermitian matrices

Vector fields form a Lie algebra, $T_1 G$ gives vector fields ..... $\longrightarrow$ $T_1 G = \mathfrak{g}$ Lie algebra :

- Exponential map A matrix $\in Mat_n$ $\Rightarrow$

$\gamma(t) = e^{tA}$ matrices depending on t, $\gamma'(t) = A\,\gamma(t)$

$\gamma(t_1)\,\gamma(t_2) = \gamma(t_1 + t_2)$: homomorphism $\mathbb{R} \to GL_n\mathbb{R}$

Conversely unique solution of $f'(t) = A\,f(t)$ with $f(0) = 1$ $\Rightarrow$ $f(t) = e^{tA}$! existence & uniqueness of solutions to ODE.

$\exp : M_n\mathbb{R} \to GL_n\mathbb{R}$ bijective near 0.

Theorem $T_1 G \longleftrightarrow$ 1-parameter subgroups for any G

e.g. $\det(e^{tA}) = e^{tr(tA)}$ so $\det = 1 \Rightarrow tr = 0$
Lie$(SL_n\mathbb{R}) = \mathfrak{sl}_n\mathbb{R}$.

SU_2 : $T_1 SU_2 = \mathfrak{su}_2$ = imaginary quaternions $\mathbb{R}^3$.
$u \in \mathbb{R}^3$ unit quaternion, $u^2 = -1$.
$\exp(tu) = \cos t + u \sin t$ rotations around u

$\Rightarrow$ exp surjective for SU_2 ... in fact for *any* compact Lie group.

Lie algebra structure on $T_1 G = \mathfrak{g}$:

$$\exp(A)\exp(B) \neq \exp(A+B) \text{ in general!}$$

Error is measured by Lie bracket, to second order.

Namely $\exp(tA)\exp(tB) = \exp(tA + tB + \frac{1}{2}t^2[A,B] + \dots$

Taylor series at $A = B = 0$.

Note ~~exp B=0~~

Another way to say this: multiplication on G to first order is addition, to second order is bracket.

Or: take $ghg^{-1}h^{-1}$ 

for g, h close to Id this will be close to id, so one param subgroups have operation on them, Lie bracket.

<u>Lie's Theorem</u> There is a map $G \longmapsto \mathfrak{g} = \mathrm{Lie}\, G$ from Lie groups to Lie algebras, and gives <u>bijection</u> $\left\{\begin{array}{c}\text{connected simply connected} \\ \text{Lie groups}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Lie} \\ \text{algebras}\end{array}\right\}$

+ homomorphisms also in bijection

[don't feel $SU_2 \to SO_3$, $SL_2\mathbb{C} \to PSL_2\mathbb{C}$ etc

$\widetilde{SL_2\mathbb{R}} \to SL_2\mathbb{R}$]

So $\mathfrak{g} \to \mathfrak{gl}_n \iff G \to GL_n$ for G simply connected.

Note Any f.d. Lie algebra arises from a Lie group!

The Stars

$GL_2\mathbb{C} = \{A: \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \det A = ad-bc \neq 0\}$

$\cup$

$SL_2\mathbb{C} : \{A \in GL_2\mathbb{C} : \det A = 1\}$

$\cup \quad \xrightarrow{2:1} PSL_2\mathbb{C} = SL_2\mathbb{C}/\pm Id = GL_2\mathbb{C}/\mathbb{C}^* = \{\begin{pmatrix} a & \\ & a \end{pmatrix}\}$

$SU_2 = \{A \cdot \bar{A}^t = Id, \det = Id\} = \{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1\}$

$= \{$unit quaternions $t + xi + yj + zk$, $t^2 + x^2 + y^2 + z^2 = 1\}$

$z + wj \longleftrightarrow \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$

$\simeq S^3 \searrow$

SO_3: ball radius π, ends identified

SO_3: $\quad (x + yi \longleftrightarrow \begin{pmatrix} x & y \\ -y & x \end{pmatrix})$

Conjugation: $SU_2 \circlearrowright \mathbb{H} \simeq \mathbb{R}^4 \supset \mathbb{R}^3 = \{xi + yj + zk\}$ preserved, acts orthogonally, $v \mapsto gvg^{-1}$

$u \in \mathbb{R}^3$ unit vector $\Rightarrow$ $g = \cos\frac{\theta}{2} + u \sin\frac{\theta}{2}$ unit quaternion, acts on $\mathbb{R}^3$ as rotation by θ

$PSL_2\mathbb{C} \simeq SO^+_{1,3}$ Lorentz group $\subset SL_4\mathbb{R}$
preserves $t^2 - x^2 - y^2 - z^2$
& direction of time

via $v = (t, x, y, z) \longmapsto A = \begin{pmatrix} t+x & y-iz \\ y+iz & t-x \end{pmatrix}$ & conjugation action

$\|v\| = \det A$ so action preserves norm.

$PSL_2\mathbb{C}$ acts on Riemann sphere S^2 via

$z \longmapsto \frac{az+b}{cz+d}$, $SU(2) \hookrightarrow SL_2\mathbb{C}$ acts as orthogonal transformations

$\twoheadrightarrow SO(3) \hookrightarrow PSL_2\mathbb{C}$

$\frac{az + b}{-\bar{b}z + \bar{a}}$

S^2 = sphere of positive light rays in $\mathbb{R}^4$
(Minkowski)
$\|v\| = 0 \,/\, \mathbb{R}^+$, Celestial sphere naturally conformal & $SO^+_{1,3}$ acts holomorphically

$$\begin{array}{ccccc} & & SL_2\mathbb{C} & & \\ & \rotatebox{45}{$\subset$} & & \rotatebox{-45}{$\supset$} & \\ SU(2) & & & & SL_2\mathbb{R} \\ & \rotatebox{-45}{$\supset$} & & \rotatebox{45}{$\subset$} & \\ & & SO(2)=U(1)=S^1 & & \end{array}$$

$SO(2) = U(1) = S^1$: $SU(2) \cap SL_2\mathbb{R} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ $\quad |a|^2 + |b|^2 = 1 = \det A$

~~Stabilizer of $SL_2\mathbb{C}$ on $S^2 \cong \mathbb{CP}^1$~~

$\cancel{\frac{az+b}{cz+d} \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix}}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} aw_1 + bw_2 \\ cw_1 + dw_2 \end{pmatrix}$ on $\mathbb{C}^2$

Up to scalar ($\mathbb{CP}^1$) : if $w_2 \neq 0 \Rightarrow$
replace $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ by $\begin{pmatrix} z = \frac{w_1}{w_2} \\ 1 \end{pmatrix}$, $\quad z \longmapsto \frac{az+b}{cz+d}$

$z = \infty \longleftrightarrow w_2 = 0 \longleftrightarrow \begin{pmatrix} * \\ 0 \end{pmatrix}$.

Stabilizer of $z = \infty \Leftrightarrow$ of $\begin{pmatrix} * \\ 0 \end{pmatrix}$ is $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$
Borel

$\Rightarrow \mathbb{CP}^1 = SL_2\mathbb{C} / B$

SU_2 acts transitively as well, stabilizer is
$B \cap SU_2 = U(1)$ $\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}$ $\quad |a| = 1$ $(\bar{a} = a^{-1})$

$S^2 = SU_2 / U(1)$: Hopf $S^1 \rightsquigarrow S^3$

$$\begin{array}{c} S^3 \\ \downarrow \\ S^2 \end{array}$$

$SL_2\mathbb{R} \curvearrowright \mathbb{P}^1$ holomorphically, not transitively

... preserves $\mathbb{RP}^1, \mathbb{H}, \mathbb{H}^-$.

$\mathbb{RP}^1 = SL_2\mathbb{R}/B_{\mathbb{R}} = \begin{pmatrix} *_{\mathbb{R}} & *_{\mathbb{R}} \\ 0 & *_{\mathbb{R}} \end{pmatrix}$

$\mathbb{H} = SL_2\mathbb{R}/\mathrm{Stab}(i) \quad : \quad \frac{ai+b}{ci+d} = i$

all $\in\mathbb{R}$, $ai+b = di-c \Rightarrow a=d,\ b=-c$ $\quad \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = U(1)$

$\mathbb{H} = SL_2\mathbb{R}/U(1)$

B_n D series

$\mathbb{R}$ $\quad SO_n = (AA^t = Id)$ $\quad$ ~~rows~~ columns orthonormal basis ~~perpendicular~~ $\leadsto$ preserve.

A series

$\mathbb{C}$ $\quad SU_n = (A\bar{A}^t = Id)$ $\quad$ columns ortho basis for $(\,,\,)$ $\quad v\cdot\bar{w}^t$

$\mathbb{H}$ $\quad Sp_n = \left(A\begin{pmatrix} & I \\ -I & \end{pmatrix}A^t = \begin{pmatrix} & I \\ -I & \end{pmatrix}\right)$, preserves inner product $(\quad)\begin{pmatrix} & I \\ -I & \end{pmatrix}(\quad)$

" C_n series

$\mathbb{O}$ $\quad E_6, E_7, E_8, F_4, G_2$

Abelian

Representations of $\mathbb{T} = S^1 = U(1) = SO(2)$

compact abelian $\subset$ compact

$\cap$ $\quad$ $\cap$

abelian $\subset$ reductive

Schur's lemma G group, (fin dim) V_1, V_2 <u>irreps</u> $\Longrightarrow$ any G-map $f: V_1 \to V_2$ is either zero or an isom.

Any G-map $f: V \to V$ is $\lambda \, Id$ $\lambda \in \mathbb{C}$

<u>Proof</u> $\ker f$ $\operatorname{Im} f$ invariant subspaces $\Rightarrow$ must be 0 or everything.

Any eigenspace $\ker(f - \lambda Id)$ invariant $\Rightarrow$ any nonempty eigenspace $= V$ so f acts by λ on V ∎

<u>Corollary</u> Any (fin dim) irrep of G <u>abelian</u> is one-dimensional, $G \longrightarrow \mathbb{C}^* = GL_1(V)$ (character)

Pf: $\rho(g)$ is a G-map for all $g \in G$! ∎

<u>Prop</u> Any f.d. rep of a <u>compact</u> Lie group is unitary $G \to U(n) \subset GL_n\mathbb{C}$! preserve hermitian inner product.

Proof: average.

So to study fin dim reps of $\mathbb{T}$, just need to study $\mathbb{T} \longrightarrow U(1) = \mathbb{T}$.

These are functions $\chi(\theta)$ of norm one with $\chi(\theta_1 + \theta_2) = \chi(\theta_1)\chi(\theta_2)$: $\chi_n(\theta) = e^{2\pi i n \theta}$ $= \cos 2\pi\theta + i \sin 2\pi\theta$

Hmm... reps as functions on G ... in fact L^2 fns ... let's study "all" functions on G

$H = L^2(S^1)$ Hilbert space, measure $\int d\theta = 1$.
$\mathbb{T}$ acts on H: $(\alpha \cdot f)(\theta) = f(\theta - \alpha)$,
unitary representation.
Let's decompose $H \leftrightarrow$ diagonalize all α's simultaneously
Note $\alpha \cdot \chi_m(\theta) = e^{2\pi i m(\theta-\alpha)} = \chi_{-m}(\alpha) \cdot \chi_m(\theta)$
so $\mathbb{C}\chi_m$ is in χ_{-m} - subrepresentation: vectors on which $\mathbb{T}$ acts via χ_{-m}.

<u>Fourier series</u>: $H = \widehat{\bigoplus \mathbb{C}\chi_{-m}}$ completed orthogonal direct sum.

Namely $f(\theta) = \sum_{m \in \mathbb{Z}} \hat{f}(m) \chi_{-m}$

$\hat{f}(m) = \int f(\theta) e^{2\pi i m \theta} d\theta = \langle f, \chi_{-m} \rangle$
component of f in χ_m direction.

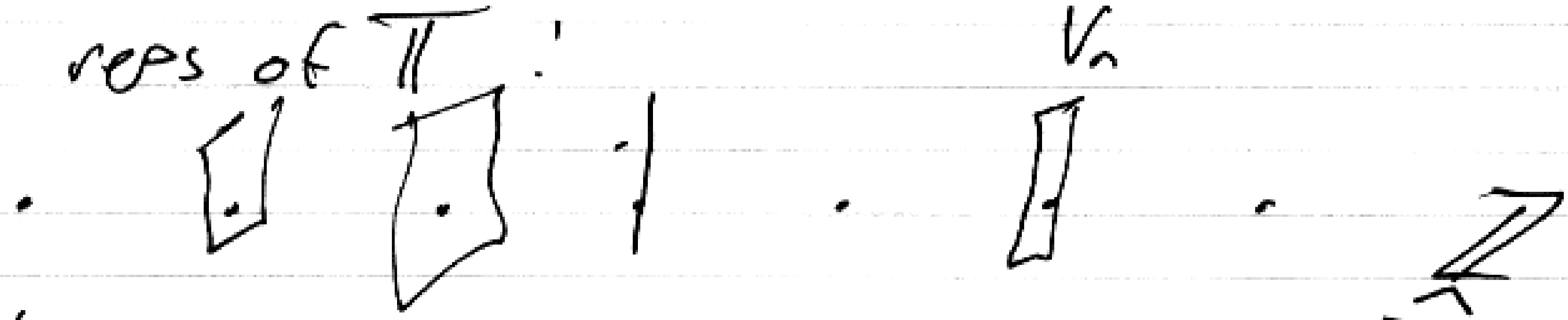
$\Rightarrow \quad L^2(\mathbb{S}^1) \simeq L^2(\mathbb{Z}) = \ell^2$

More generally, given <u>any</u> rep of $\mathbb{T}$, (V, π)
get Fourier decomposition:
$v \in V \longmapsto v_n = \int_{\mathbb{T}} (\pi(\theta) \cdot v) e^{-2\pi i n \theta} d\theta \in V$

$\pi(\theta) \cdot v_n = \chi_n(\theta) \cdot v_n \in \underbrace{\chi_n\text{ - component of } V}_{V_n}$

$V = \widehat{\bigoplus V_n}$: ie each V_n is a closed subspace, every v has a <u>unique</u> convergent expansion $v = \sum v_n$. <u>Don't know which topology in general!!</u> [eg C^∞, L^1, $\mathcal{S}$ etc on $\mathbb{T}$!!

i.e. reps of $\mathbb{T}$:



Vector space valued function on $\mathbb{Z}$ $= \hat{\mathbb{T}}$

<u>Admissible</u>: all V_n finite dimensional.

Have $C^\infty(S^1)$: $\sum a_n \chi_n$, $a_n \to 0$ faster than any polynomial

$C^\omega(S^1)$: $\sum a_n \chi_n$; $a_n \to 0$ exponentially ($K^{|n|} a_n$ bounded $K > 1$)

Dually have distributions / hyperfunctions: $a_n \to \infty$ at most polynomially / exponentially.

<u>Algebraic part</u>: $\bigoplus V_n \subset V$.

For $L^2(S^1)$: exactly $\bigoplus \mathbb{C} e^{2\pi i n x} = \bigoplus \mathbb{C} z^n$

$z^n : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ algebraic representations of $\mathbb{C}^*$.

$\mathbb{C}^*$ = complexification of $U(1)$:
define as group with functions $\bigoplus \mathbb{C} z^n$
or as Spec $\bigoplus \mathbb{C} z^n$.

<u>Fourier transform</u> Pass from $\mathbb{T}$ to $\mathbb{R}$.

- Not all characters unitary: $\mathbb{R} \to \mathbb{C}^*$ are $x \mapsto e^{itx} = \chi_t(x)$, unitary iff $t \in \mathbb{R}$.

(Unitary) characters of $\mathbb{R}$ not in L^2: $|e^{itx}| = 1$ ...

- Have indecomposable, not irreducible reps!

Two approaches: analytic & algebraic.

Fourier transform: $\hat{f}(t) = \int_{\mathbb{R}} f(x) \chi_{-t}(x)\, dx$

$y \in \mathbb{R} \Rightarrow \tau_y$ translation $\tau_y f(x) = f(x-y)$

$(\tau_y f)^\wedge (t) = \chi_t(y) \hat{f}(t)$: "diagonalize τ_y operators simultaneously" - make translation into multiplication. ... $\hat{f}$ coefficients for writing f as "continuous linear combination" of $\chi_t(x)$

Lie algebra version: $\left(\frac{d}{dy}\right)^\wedge = t \cdot \_$

How to make sense of this?

$\mathcal{S}(\mathbb{R})$ Schwartz space: f C^∞ of rapid decay - f and all derivatives decay faster than any polynomial.

$\wedge : \mathcal{S}(\mathbb{R}) \xrightarrow{\sim} \mathcal{S}(\mathbb{R})$

$\Rightarrow \wedge : \mathcal{S}(\mathbb{R})^* \xrightarrow{\sim} \mathcal{S}(\mathbb{R})^*$ <u>tempered distributions</u>

- e.g. δ-function $\delta_s(f)$: $\delta_s(f) = f(s)$

& $\chi_t(x)$ $(t \in \mathbb{R})$: $\chi_t(f) = \int f \chi_t\, dt$.

In fact $\chi_t^\wedge = \delta_t$.

So writing $f(x) = \frac{1}{2\pi} \int \hat{f}(t) \chi_t(x)$

multiple of δ_t in $\hat{f}$, ie $\hat{f}(t)$, is how much χ_t plays into decomposition

Plancherel Another way: $\wedge : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ isometry (define first on $L^1 \cap L^2$, extend by continuity)

$$L^2(\mathbb{R}) = \int_{\mathbb{R}} \mathbb{C}\chi_t \, dt \quad \text{direct integral}$$

Abelian 4$\frac{1}{3}$

Fourier transform as representation theory of $\mathbb{R}$:

$f(x) = \int \hat{f}(t)\, \chi_t(x)\, dt$ continuous linear combination of χ_t's

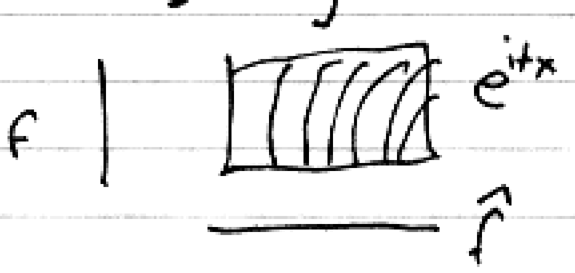
χ_t's are characters of $\mathbb{R}$: $\tau_y \chi_t(x) = \chi_{-t}(y)\, \chi_t(x)$

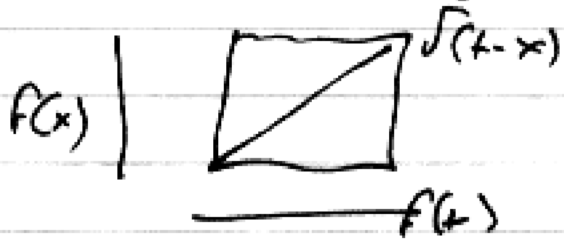
$\hat{f}$ are coefficients in decomposition into χ_t's –

e.g. $\hat{\chi_s}(t) = \delta_s(t)$ only need a single χ_t to write χ_t:

$\chi_s(x) = \int \delta_s(t)\, \chi_t(x)\, dt$ More generally $f(x) = \int f(t)\, \delta_x(t)\, dt$

$= \int f(t)\, \delta(t-x)\, dt$





Any fn linear combo of its values: δ-fns form "standard" basis of functions

$(\tau_y f)^\wedge(t) = \chi_t(y)\, \hat{f}(t)$: in e^{ixt} basis, τ_y is multiplication by $\chi_t(y)$ on t'th "entry".

Infinitesimal version:

where makes sense! .... $(\frac{\partial}{\partial x} f)^\wedge(t) = it\, \hat{f}(t)$ $\quad it = \frac{\partial}{\partial y} e^{ity}\Big|_{y=0}$

$\wedge$ interchanges differentiation & multiplication $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} \tau_y \Big|_{y=0}$

<u>Group algebra</u> G finite group, $\mathbb{C}[G]$ has basis δ_g. Make into algebra:

$\delta_g \cdot \delta_{g'} = \delta_{gg'}$. <u>Noncommutative</u> if G is:

$\delta_{g'} \delta_g = \delta_{g'g} \neq \delta_{gg'}$ in general.

For general $f, h \in \mathbb{C}[G]$: $f = \sum f(g)\, \delta_g$ $\quad h = \sum h(g)\, \delta_g$

$fh(g) = \sum_{g'g''=g} f(g')\, h(g'') = \sum_{g' \in G} f(g')\, h((g')^{-1} g)$

convolution

Abstractly: extend $G \times G \to G$ to
$\mathrm{Meas}(G) \times \mathrm{Meas}(G) \to \mathrm{Meas}(G)$
... opposite direction to functions.

G locally compact group: have convolution on measures or distributions ~~$\mu * \nu$~~ $= \int \mu \nu(g x)$

... fix measure on G, convolve functions f, h
$$f * h(g) = \int_G f(g') h((g')^{-1} g) \, dg'$$

δ-measures $\delta_g * \delta_{g'} = \delta_{gg'}$.

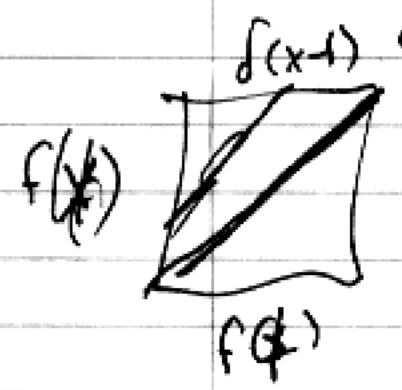
When does $*$ make sense? e.g. $C_c(G)$ continuous functions with compact support. or $L^1(G)$ integrable functions

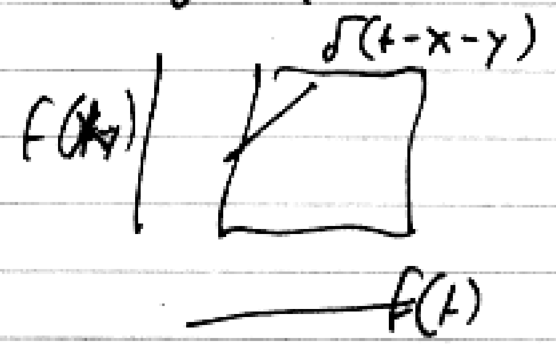
$\mathbb{R}$: $(f*h)^\wedge(t) = \hat{f}(t) \cdot \hat{h}(t)$

($\mathbb{C}[G]^*$ act by multiplication operators on $\mathrm{Fun}(\hat{\mathbb{R}})$).

e.g. $\delta_y * f(x) = \int \delta_y(t) f(x-t) dt = f(x-y)$

$\delta_0 * f = f$ identity operator.





$\pi: G \circlearrowright V$ finite group action $\Rightarrow$

$\mathbb{C}[G]$ acts: $\pi(\delta_g) = \pi(g) \in \mathrm{End}\, V$

$$\pi(f) v = \pi\left(\sum f(g) \delta_g\right) = \sum f(g) \pi(\delta_g) v = \sum f(g) \pi(g) \cdot v$$

Continuous: $\pi(f) v = \int_G f(g) \pi(g) \cdot v \, dg = \int_G f(g) g \cdot v \, dg$

So (π, V) rep of G Lie group on (Hausdorff loc convex complete) t.v.s. V

$$\Rightarrow C_c(G), L^1(G) \circlearrowright V \text{ via}$$

$\pi(f)v := \int_G f(g)\, g v \, dg$. associative algebras, cont. action

$\rightsquigarrow$ C^* algebra: Banach algebra with conjugate linear involution $x \longmapsto x^*$ s.t. $(xy)^* = y^* x^*$

$$\|x^*\| = \|x\|, \quad \|x x^*\| = \|x\|^2.$$

Direct integral of reps:

X, μ measure space $\mathcal{H}$ $x \longmapsto H_x$ "measurable assignment of vector spaces"

e.g. locally trivial, or specify class of measurable sections $v(x)$ s.t. $\|v(x)\|$ measurable function, countable set $v_i(x)$ span dense subspace etc.

$\Rightarrow$ look at L^2 sections of $\mathcal{H}$ $\Rightarrow \int H(x)\, d\mu$.

Projection valued measure: $H(x) = \operatorname{Im} P_x$ projection at x.

Any rep of $\mathbb{R}$ $V = \int_{\hat{\mathbb{R}}} V(t)\, dt$ .... $\rightsquigarrow$ vN algebra theory...

Convolution & Fourier:

$$\langle g, e^{ix y} f \rangle = \int_{\mathbb{R}^n} e^{-ixy} f(x) g(x)\, dx = (f \cdot g)^\wedge$$

$$\langle \hat{g}, \widehat{e^{iy} f} \rangle = \int \left(\int e^{-\lambda x + iyx} \bar{f}(x)\, dx \right) \cdot \hat{g}(\lambda)\, d\lambda$$

$$= \int \hat{f}(y - \lambda)\, \hat{g}(\lambda)\, d\lambda = \hat{f} * \hat{g}$$

Note : $t \cdot \delta_\tau = \tau \delta_\tau$ diagonal

BUT $t \cdot \delta'_\tau = \tau \delta'_\tau - \delta_\tau$:

$\delta'_\tau((t-\tau) f(t)) = -f(\tau)$

so t acts by $\begin{pmatrix} \tau & -1 \\ & \tau \end{pmatrix}$ on $\begin{pmatrix} \delta_\tau \\ \delta'_\tau \end{pmatrix}$

Higher derivatives: Jordan blocks...

~~Algebraic version~~:

L^2 spectral theorem POV: $L^2(\mathbb{R})$ carries projection-valued measure on $\hat{\mathbb{R}}$:

$\boxed{\text{char}_U}$

measurable $U \subset \hat{\mathbb{R}} \to$ project onto $L^2(U) \subset L^2(\mathbb{R})$ = multiply by $\cancel{1_U}$

$$P_{U_1 \cup U_2} = P_{U_1} + P_{U_2} - P_{U_1 \cap U_2}. \quad P_{U_1 \cap U_2} = P_{U_1} \cdot P_{U_2}$$

commuting with action of $\mathbb{R}$

More generally any unitary rep H of $\mathbb{R} \Rightarrow H$-projection valued measure on $\hat{\mathbb{R}}$ giving decomposition

$x \in \mathbb{R}$ $\quad \rho(x) = \int e^{itx} \, dP$.

$$\text{- analog of } \quad \rho(x) \cdot v = \sum_{t \in \mathbb{R}} e^{itx} \underset{\substack{\downarrow \\ \text{projection on eigenspace}}}{v_t}$$

$\Rightarrow$ Direct integral...

<u>Pontrjagin - van Kampen duality</u>

G locally compact abelian group $\Rightarrow$

$\hat{G}$ = unitary characters $\mathrm{Hom}(G, U(1))$ is a locally compact abelian group

$x \in G \to x(\hat{x}) = \langle x, \hat{x} \rangle$ function on $\hat{G}$ & vice versa

$\Rightarrow$ G acts on $L^2(\hat{G})$ by multiplication operators

Abelian
6

Fourier transform $L^2(G) \to L^2(\hat{G})$:

$$\hat{f}(\hat{x}) = \int f(x) \langle x, \hat{x} \rangle \, dx$$

$$f(x) = \int \hat{f}(\hat{x}) \overline{\langle x, \hat{x} \rangle} \, d\hat{x}$$

Haar measures, determine each other

translation $(\tau_x \cdot f)^\wedge = M_x \hat{f}$ multiplication

e.g. $\mathbb{Z}^\wedge = \mathbb{T}$

$L^\wedge = \mathbb{R}/L^\vee$

$\mathbb{R}^\wedge = \mathbb{R}$

Any unitary rep $G \curvearrowright H \Rightarrow$
H-projection valued measure on $\hat{G}$

$$\rho(x) = \int \langle x, \hat{x} \rangle \, dP$$

Note: Poisson summation

Algebra V admissible: maximal compact acts with finite multiplicity

$\mathbb{R}$ case: f.dim $\rho : \mathbb{R} \to \mathrm{Aut}\, V \supset U(V)$

differentiate: $d\rho : \mathbb{R} \to \mathfrak{gl}_n$, $\mathbb{R} \to \mathfrak{u}_n \to \mathfrak{gl}_n$

Single matrix, up to conjugation...

2/3/05

Fourier transform diagonalizes $\mathbb{R} \circlearrowright \mathrm{Fun}(\mathbb{R})$

- $(\tau_y f)^\wedge(t) = \chi_t(y) \cdot \hat{f}(t)$; analysis: which function space each makes sense on
- $(\frac{d}{dx} f)^\wedge(t) = t \hat{f}(t)$
- $(h * f)^\wedge(t) = \hat{h}(t) \cdot \hat{f}(t)$ $\quad$ ($h \in C_c(\mathbb{R})$ group alg)

Here $h * f(x) = \int_{\mathbb{R}} h(s) f(x-s)\, ds$

e.g. $\delta_a * f(x) = \int_{\mathbb{R}} \delta_a(s) f(x-s)\, ds = f(x-a)$
translation operator

$\delta_0 * f(x) = f(x)$ identity of group

$G \circlearrowright V$ action on vector space $\Rightarrow$ linearize in two ways:
- Differentiate: $\mathfrak{g} \to \mathrm{End}\, V$
- Take linear combos: $\mathrm{Fun}(G) \longrightarrow \mathrm{End}\, V$
  $f \cdot v = \int_G f(g)\, g \cdot v\, dg$

Let's call t-line $\hat{\mathbb{R}} = \{\text{homomorphisms } \mathbb{R} \longrightarrow U(1)\}$

$\mathrm{Fun}(\mathbb{R})$ is continuous direct sum of $\mathbb{C}\chi_t$, $t \in \hat{\mathbb{R}}$
$\Rightarrow L^2(\mathbb{R}) = \int_{\mathbb{R}} \mathbb{C}\chi_t\, dt$ direct integral

What does this mean? algebraically $\mathbb{R}$-module (fin-dim)
$\Rightarrow \mathbb{C}[t]$-module, "fiber" at $t \in \hat{\mathbb{R}}$: $V/(t-\lambda)\cdot V$

analytically $H = L^2(\mathbb{R})$ carries projection-valued measure over $\hat{\mathbb{R}}$: to each $U \subset \hat{\mathbb{R}}$ assign orthogonal projection P_U on $L^2(\mathbb{R}) = H$ $P_U^2 = \mathrm{Id}$,
P_U projects onto $L^2(U) \subset L^2(\hat{\mathbb{R}}) \overset{\wedge}{\simeq} L^2(\mathbb{R})$
commuting with $\mathbb{R}$ action : $\mathbb{R}$ preserves image of P_U.

$P_{U_1 \cup U_2} = P_{U_1} + P_{U_2} - P_{U_1 \cap U_2}$, $P_{U_1 \cap U_2} = P_{U_1} \cdot P_{U_2}$

In fact action of $x \in \mathbb{R}$ by translation τ_y is given as a linear combo of the P's:

analog of $\tau_y \cdot V = \sum \chi_n(y) \cdot V_n$ V_n = projection onto χ_n eigenspace

ie $\tau_y = \sum \chi_n(y) \cdot P_n$ projection onto V_n V_n

.... $\tau_y = \int_{\mathbb{R}} e^{itx} dP$.

Direct integral of representations:

$\hat{\mathbb{R}}$, dt measure space. $t \mapsto \mathcal{H}_t$ measurable assignment of vector spaces ... e.g. trivial, or have notion of measurable sections $v(t), u(t)$ s.t. $\langle v(t), u(t) \rangle$ measurable functions

$\Rightarrow$ Hilbert space $\mathcal{H} = \int_{\hat{\mathbb{R}}} \mathcal{H}_t \, dt$, have P_U measure

with action of $\mathbb{R}$: $x \in \mathbb{R}$ acts by multiplication by function $\chi_t(x)$ on $\hat{\mathbb{R}}$. - ie scalar operator on each $\mathcal{H}_t$.

Theorem Any unitary rep of $\mathbb{R}$ is of this form.

Where do Jordan blocks come from? $\underline{\quad\quad \mathsf{X} \quad\quad}$

$\delta_s'(t)$ distribution, with $\delta_s'(f(t)) = -f'(s)$

$\left(\int_{\mathbb{R}} \delta_s'(t) f(t)\, dt = -\int_{\mathbb{R}} \delta_s(t) f'(t)\, dt = -f'(s) \right)$

Calculate: $t\delta_s(f) = (tf)(s) = s \cdot f(s)$

ie $t \cdot \delta_s = s\,\delta_s$

But : $t\delta_s'(f) = -(tf)'(s) = -sf'(s) - f(s)$

so $t\delta_s' = s\delta_s' - \delta_s$.

So t acts by $\begin{pmatrix} s & +1 \\ 0 & s \end{pmatrix}$ on basis $\begin{pmatrix} \delta_s \\ -\delta_s' \end{pmatrix}$.

Similar Jordan blocks for higher derivatives.

Fourier dually: $\mathbb{R}$ acts by translation on $\begin{pmatrix} e^{isx} \\ xe^{isx} \end{pmatrix}$

$\tau_y \begin{pmatrix} e^{isx} \\ \cancel{x}e^{isx} \end{pmatrix} = \begin{pmatrix} e^{isy} & -ye^{isy} \\ & e^{isy} \end{pmatrix}$, $\frac{d}{dx}\begin{pmatrix} e^{isx} \\ -xe^{isx} \end{pmatrix} = \begin{pmatrix} s & +1 \\ 0 & s \end{pmatrix}$

<u>Pontrjagin duality</u> . G abelian (Lie) group [locally cpt topological]

$\Rightarrow$ $\hat{G} = \mathrm{Hom}(G, U(1))$ characters (unitary) is again a Lie group : two characters nearby if their values on all $g \in G$ are nearby.

e.g. $\hat{\mathbb{T}} = \mathbb{Z}$, $\hat{\mathbb{Z}} = \mathbb{T}$

$\hat{\mathbb{R}} = \mathbb{R}$ $\widehat{\text{Finite}} = \text{Finite}$

$\widehat{\mathbb{Z}/n} = \mathbb{Z}/n$. : roots of unity for generator.

$x \in G \Rightarrow x(\hat{x}) = \langle x, \hat{x}\rangle$ function on $\hat{G} \Rightarrow$

G acts on $L^2(\hat{G})$ by multiplication operators

Fourier transform : $\hat{f}(\hat{x}) = \int f(x) \langle x, \hat{x}\rangle dx$

$f(x) = \int \hat{f}(\hat{x}) \overline{\langle x, \hat{x}\rangle} d\hat{x}$

invariant measures normalized appropriately

$(\tau_x f)^\wedge = \langle x, -\rangle \cdot \hat{f}$

Theorem • $\hat{\hat{G}} = G$ • $L^2(G) \xrightarrow{\sim} L^2(\hat{G})$

• Any unitary rep $G \circlearrowright V \Rightarrow$

H-valued projection valued measure $V = \int_{\hat{G}} V_{\hat{x}} \, d\hat{x}$

sections of bundle on $\hat{G}$.

Trace & Poisson Summation

A $n\times n$ matrix, e'values (repeated) λ_i

$\Rightarrow \quad \sum A_{ii} = \sum \lambda_i$: can evaluate tr abstractly or by diagonalizing

Geometrically : $v_1 \cdots\cdots v_n$

$v \in \mathbb{C}^n$ as function on n points X

$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

$A \cdot v = \begin{pmatrix} \overline{} \\ -\sum a_{ij} v_j - \\ \overline{} \end{pmatrix}$; $a_{i1}\, a_{i2} \cdots a_{in}$

$v_1 \cdots v_n$

~~Spectrally~~ :

$\Leftrightarrow$ trace is $\sum \langle A e_i, e_i \rangle$

e_i component of $A \cdot e_i$

$A = A(i,j)$ function of two variables, $X \times X$

<u>More generally</u> : $K(x,y) \in C^\infty(X \times X)$ X compact manifold

$\Rightarrow$ operator $\mathcal{O}_K \cdot f = \int K(x,y)\, f(y)\, dy$

<u>Def</u> $\quad \mathrm{Tr}\, \mathcal{O}_K = \int_X K(x,x)\, dx$ add up diagonal entries

... write function in basis of δ-functions, take $\sum \langle \mathcal{O}_K \cdot e_i, e_i \rangle$

... analysis : trace-class operator...

Poisson summation : $\mathbb{R} \curvearrowright \underbrace{L^2(\mathbb{R}/\mathbb{Z})}_{=\mathbb{T}}$ by translation

We know $L^2(\mathbb{R}/\mathbb{Z}) = \bigoplus_{2\pi\mathbb{Z}} \mathbb{C}\chi_n$. one dimensional.

$f \in C(\mathbb{R})$: acts on $\bigoplus \mathbb{C}\chi_n$ as $\begin{pmatrix} \ddots & & \\ & \hat{f}(n) & \\ & & \ddots \end{pmatrix}$

so trace is $\sum_{2\pi\mathbb{Z}} \hat{f}(n)$.

But can also calculate trace "geometrically" as "sum of diagonal entries" :

Let $\tilde{f}(\theta) = \sum_{\mathbb{Z}} f(\theta + n)$ for $\theta \in [0,1)$ ~~or~~ $\mathbb{T}$.

ie $\int_{\mathbb{R}\to\mathbb{Z}} f = \tilde{f}$.

Then calculate action of $\dot{f} \in C(\mathbb{R})$ on $h \in$ ~~C^∞~~ $L^2(\mathbb{R}/\mathbb{Z})$:

$$f * h(\theta) = \int_{\mathbb{R}} f(x)\, \tau_x h(\theta)\, dx$$

$$= \int_{\mathbb{R}} f(x)\, h(\theta - [x])\, dx = \int_{\mathbb{T}} \tilde{f}(\alpha)\, h(\theta - \alpha)\, d\alpha$$

$\alpha = x \bmod \mathbb{Z}$

$$= \tilde{f} * h(\theta)$$

But this is an integral operator: $K(x,y) = \tilde{f}(x-y)$

$\tilde{f} * h = \mathcal{O}_K \cdot h$.

So $\operatorname{tr}\langle f \rangle = \int_{S^1} \tilde{f}(x-x)\,dx = \hat{f}(0) = \sum_{\mathbb{Z}} f(n)$

$\Rightarrow \sum f(n) = \sum \hat{f}(2\pi n)$. ∎

Note: $\operatorname{Tr}_{L^2(\mathbb{R}/\mathbb{Z})} : f \longmapsto \sum f(n)$

$(\sum \delta_n)^\wedge = \sum \delta_{2\pi n}$ ie $\operatorname{Tr}_{L^2(\mathbb{R}/\mathbb{Z})} = \sum \delta_n$ as distribution!

Induced rep: G, H finite, $V = \operatorname{Ind}_H^G \mathbb{C} = \mathbb{C}[G/H]$

$\operatorname{Tr}_V g = \begin{cases} 1 & g \in H \\ 0 & \text{otherwise} \end{cases}$ ie $\chi_V = \delta_H$.

$\Rightarrow$ <u>distributional character</u>.

[$G \supset L$ lattice, $\hat{G} \supset L^\vee$ annihilator: $\check{\ell}(\ell) = 1\ \forall \ell$.

$\hat{L} = \hat{G}/L^\vee$ $\quad L^\vee = (G/L)^\wedge$ $\quad \sum_{L} f(\ell) = \sum_{L^\vee} \hat{f}(\check{\ell})$

• Maximal compact subgroups

Theorem G Lie group, $(|G/G^0| < \infty) \Rightarrow$
$\exists$ maximal compact subgroups, any two are conjugate ($\Rightarrow$ Any compact subgroup conjugate to subgroup of $K < G$ max cpt) &
$G \simeq K \times \mathbb{R}^m$ some m homeomorphism

Pf for $GL_n\mathbb{R} \Rightarrow K = O_n$ orthogonal / $GL_n\mathbb{C} \Rightarrow K = U_n$

Show any compact subgroup preserves an inner product, by integration:

$$\langle \xi, \eta \rangle_K = \int_K \langle k\xi, k\eta \rangle dk$$

Digression What does $\int_K$ mean? for any fn $f : K \to V$ continuous, valued in a (locally convex, complete) vector space $\Rightarrow \int_K f(k)\, dk \in V$:

$\int_K : C(K; V) \to V$ s.t.

i. $\int_K f(k) dk = c$ if $f(k) = c$ constant fn
($\int_K dk = 1$)

ii. $\int_K f(gk)\, dk = \int_K f(kg)\, dk = \int_K f(k)\, dk \quad \forall g \in K$

in fact convex: iii. f takes values in convex $C \subset V \Rightarrow$
(measure positive) $\int_K f \in C$.

How? Need volume form $\omega_n = dx_1 \wedge \ldots \wedge dx_n$.

Pick it at one point $1 \in K$, use left translation to define it everywhere, normalize.

Unimodular: apply left translation: $\int f(kg)\, dk$
$\mu = 1$ since $\mathbb{R}_+^*$ has no compact subgroup. $\mu : K \to \mathbb{R}_+^*$

e.g. $K = SU_2 \simeq S^3$ usual volume form.

On $GL_n\mathbb{R}$: $\int_G f = \int \frac{f(A)}{|A|^n}\, dA$ is left & right invariant integral ... but $\int 1\, dg$ diverges

e.g. $\mathbb{C}^\times$ $\frac{dx}{x}$ is invariant, dx isn't.

<u>Pf continued</u> Take $\langle , \rangle_K$ and find orthonormal basis u_i by Gram-Schmidt : v_i basis $\to u_i$ ortho

$$v_1 = \|v_1\| u_1$$

$$v_2 = \left(-\text{proj on } v_1\right) + \frac{u_2}{\text{norm}} \qquad u_2 = \frac{v_2 - \text{proj onto } u_1}{\text{norm}}$$

$$= \lambda_{12} u_1 + \lambda_{22} u_2$$

$$v_3 = \lambda_{13} u_1 + \dots + \lambda_{33} u_3$$

$$\Rightarrow \quad g = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \begin{pmatrix} \ddots & 0 \\ \lambda & \ddots \end{pmatrix}$$

$GL_n\mathbb{C}$: $\forall$ unitary lower triang

$GL_n\mathbb{C} \ni g = ub \quad u \in U_n \quad b \in B_-$ with positive real diagonal entries

$GL_n\mathbb{C} = U_n B_-^{>}(\mathbb{C})$

~~$GL_n\mathbb{R} = O_n B_-^{>}(\mathbb{R})$~~

Real case : polar decomposition $g = p\,o$ p pos-def symmetric, o orthogonal :

$p = (gg^t)^{\frac{1}{2}}$! $\sqrt{\ }$ of pos-def symmetric

$o = p^{-1} g$

K preserves $\langle , \rangle_K$, after change of basis to orthonormal basis will be $\subset O_n \,/\, U_n$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "g"] \arrow[d, "k"'] & \mathbb{R}^n \arrow[d, "g^{-1}kg"] \\ \mathbb{R}^n \arrow[r, "g"'] & \mathbb{R}^n \end{tikzcd}$$

Example: $SL_2\mathbb{R} \supset SO_2$ max compact. $\simeq \mathbb{T}$

$SL_2\mathbb{R} \curvearrowright \mathbb{H} = SL_2\mathbb{R}/SO_2$ $\quad SL_2\mathbb{R} \simeq$ unit tangent bundle of $\mathbb{H}$ $\simeq D \times S^1$

$\widetilde{SL_2\mathbb{R}} \simeq D \times \mathbb{R}$ universal cover.

$PSL_2\mathbb{C} \curvearrowright \mathbb{P}^1 = SL_2\mathbb{C}/B$ $\quad$ follows from $GL_n\mathbb{C} = U_n B$

$SU_2 \curvearrowright S^2 = SU_2/\mathbb{T}$ $\quad GL_n/B = U_n/(U_n \cap B = \mathbb{T}^n)$

Maximal tori

K compact connected $\Rightarrow$ any $k \in K$ is conjugate to an element of a maximal torus $T \simeq \mathbb{T}^n$ & any connected abelian subgroup conjugate to a subgroup of T $\Longrightarrow$ any two max tori conjugate.

e.g. $U_n \supset T$ = diagonal matrices $\simeq \mathbb{T}^n$

- ie any unitary matrix diagonalizable.

Same for real orthogonal matrix

Our case: $SO_3 \supset SO_2$ maximal torus:

every element ~~of a~~ SO_3 is rotation about some axis ...

... $\Longleftrightarrow$ "every element of SO_3 acting on S^2 ~~...~~ SO_3/SO_2 has a fixed point : follows from topology

... hairy ball theorem (connect to fixed-point theory)

Noncompact/Complex groups: false

Complexification Lie algebra: $\mathfrak{g}_\mathbb{R} \to \mathfrak{g}_\mathbb{C} = \mathfrak{g}_\mathbb{R} \otimes \mathbb{C}$.

e.g. $\mathfrak{gl}_n\mathbb{C} = \mathfrak{u}_n \otimes \mathbb{C}$: any matrix $= A + iB$ with A, B skew hermitian

$M = \cancel{\frac{M+\bar{M}^t}{2}} \cancel{2} \frac{M-\bar{M}^t}{2}$ $\quad M = \frac{M-\bar{M}^t}{2} + i\,\frac{-iM-i\bar{M}^t}{2}$

Finite-Dimensional Reps of $\mathfrak{sl}_2\mathbb{C}$

V (fdim) irrep of $\mathfrak{sl}_2 = \text{span}\{e, f, h\}$

$h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \quad e = \begin{pmatrix} & 1 \\ & \end{pmatrix} \quad f = \begin{pmatrix} & \\ 1 & \end{pmatrix}$

$[e,f] = h$
$[h,e] = 2e$
$[h,f] = -2f$

First observation: let V_λ = λ eigenspace of h, $h \cdot v = \lambda v$

$v \in V_\lambda \quad h(e\cdot v) = e(hv) + ([h,e])\cdot v = \lambda(ev) + 2(ev)$
$= (\lambda+2)\, ev$

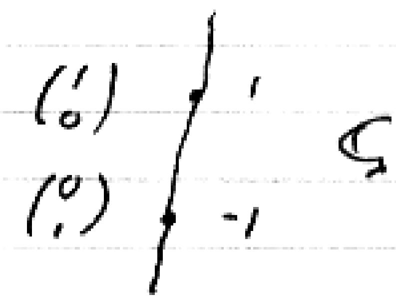
$\boxed{e : V_\lambda \to V_{\lambda+2}}$

$h(fv) = f(hv) + ([h,f])\cdot v = (\lambda-2)\, fv$

$\boxed{f : V_\lambda \to V_{\lambda-2}}$

e.g. $\mathbb{C}^2 = \begin{pmatrix}1\\0\end{pmatrix} \oplus \begin{pmatrix}0\\1\end{pmatrix}$

$0 \leftarrow V_1 \underset{f}{\overset{e}{\rightleftarrows}} V_{-1} \rightarrow 0$



Suppose $V_\lambda \neq 0$, $V_{\lambda+2} = 0$: finite-dimensionality (or just assume)

$0 \neq v_\lambda \in V_\lambda$

$\Rightarrow \boxed{\begin{array}{l} e\cdot v_\lambda = 0 \\ h\cdot v_\lambda = \lambda v_\lambda \end{array}}$ can't raise : highest weight vector, weight λ

Let $v_{\lambda-2s} = \underbrace{f\cdot f\cdot \ldots \cdot f}_{s} \cdot v_\lambda = \pi(f)^s \cdot v_\lambda$

Note don't have powers in a Lie algebra, but do in any rep...

Check $h\, v_{\lambda-2s} = h f f \ldots f \cdot v_\lambda = f h \underbrace{f\ldots f}_{s-1} v_\lambda + [h,f]\underbrace{f\ldots f}_{s-1} v_\lambda$

$= f h f \ldots f v_\lambda - 2\underbrace{f\ldots\ldots f}_{s} v_\lambda$

$= f f h \underbrace{f\ldots f}_{s-2} v_\lambda - 4\underbrace{f\ldots f}_{s} v_\lambda = \ldots = \underbrace{f f\ldots f}_{s} h v_\lambda - 2s\, f\ldots f v_\lambda$

$= (\lambda - 2s)\, f\ldots f v_\lambda = (\lambda-2s)\, v_{\lambda-2s}$

weight $\lambda - 2s$
= eigenvalue.

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$$\begin{array}{c} 0 \\ V_\lambda \\ f \downarrow \uparrow e \\ V_{\lambda-2} \\ f\downarrow \uparrow e \\ V_{\lambda-4} \\ \vdots \\ f \downarrow \\ V_m \\ 0 \end{array}$$

At each stage, e takes us back up!

Claim: $e f^{n+1}(v) = f^{n+1} e v + (n+1) f^n (h-n) v$

Proof: induction on n. $n=0$: $ef = fe + h$
$hf = f(h-2)$

assume for n. $e f^{n+2} v = (f^{n+1}e + (n+1) f^n (h-n)) f v$

$= f^{n+2} e v + f^{n+1} h v + (n+1) f^{n+1}(h-n-2) v$

$= f^{n+2} e v + (n+2) f^{n+1}(h-n-1) v$ ∎

ie. $e v_{\lambda-2s} = e f^s v_\lambda = \underbrace{f^s e v_\lambda}_{0} + s f^{s-1}(h-(s-1)) v_\lambda$

$= s(\lambda - s + 1) v_{\lambda-2s+2}$

Fin dim $\Rightarrow$ eventually have $f^m v_\lambda \neq 0$, $f^{m+1} v_\lambda = 0$

... ie. $\boxed{\begin{array}{l} f\, v_{\lambda-2m} = 0 \\ h\, v_{\lambda-2m} = (\lambda-2m) v_{\lambda-2m} \end{array}}$ lowest weight vector.

When does this happen?

$$\begin{array}{c} 0 \\ v_\lambda \\ \vdots \\ v_{-\lambda} \end{array}$$

$0 = e f^{n+1} v_\lambda = f^{n+1} e v_\lambda + (n+1) f^n (h-n) v_\lambda$

$= (n+1)(\lambda - n) v_{\lambda-2n} = 0$

$\Rightarrow \lambda = n$. $\Rightarrow \lambda$ is an positive integer!

- complete description of finite dim reps of $\mathfrak{sl}_2 \mathbb{C}$!
... in particular note h diagonalizable:

$h = \begin{pmatrix} \lambda & & & \\ & \lambda-2 & & \\ & & \ddots & \\ & & & -\lambda \end{pmatrix}$ $f = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}$

$e = \begin{pmatrix} 0 & \lambda & & & & \\ & 0 & 2(\lambda-1) & & & \\ & & 0 & 3(\lambda-2) & & \\ & & & \ddots & \ddots & \\ & & & & 0 & (\lambda-1)2 \\ & & & & & \lambda \\ & & & & & 0 \end{pmatrix}$

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Another point of view: $SL_2\mathbb{C} \curvearrowright V$ irrep, algebraic/holomorphic

$SL_2\mathbb{C} \supset \mathbb{C}^* = \left\{\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\right\} = T$

$\boxed{\text{Non-algebraic: could get } z \mapsto z^p \bar{z}^q}$

$p-q \in \mathbb{Z}$

$\mathbb{C}^* \curvearrowright V \Rightarrow$ decomposition $V = \bigoplus_{i\in\mathbb{Z}} V_i$

On V_i: $t = \begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}$ acts by a^i ... i^{th} character.

What restriction does being a rep of alg $SL_2\mathbb{C}$ put on the <u>weights</u> $\{i\}$ that appear?

$Z_{SL_2\mathbb{C}}(\mathbb{C}^*) = \mathbb{C}^*$: centralizer is $\mathbb{C}^*$, so don't have any other matrices guaranteed to preserve the decomposition.

- look at normalizer $N(T) = \{n \in G : n^{-1}tn \in T \ \forall t \in T\}$

... matrices on $\mathbb{C}^2$ that take decomposition $\mathbb{C}^2 = \mathbb{C}_1 \oplus \mathbb{C}_{-1}$ to itself $\Rightarrow$ monomial matrices (permutation)

$\begin{pmatrix} \cdot & \\ & \cdot \end{pmatrix}$ or $\begin{pmatrix} & \cdot \\ \cdot & \end{pmatrix}$

In rep V: $n \in N$, $v \in V_i$ ie $t \cdot v = a^i v$

$\Rightarrow t(nv) = \cancel{(ntn^{-1})}$ $n t^{(n)} v$ again a scalar multiple of nv; $t^{(n)} = n^{-1}tn \in T$

i.e. n takes T weight spaces to other T weight spaces. [in general $N(H)$ preserves H & decompositions

~~what this means~~ $N(H)$ acts on H, so acts on reps of H: $\rho^n(h) = \rho(h^{(n)}) = \rho(\underbrace{n^{-1}hn}_{\in H})$ new rep.

So if $V = \bigoplus V_i$ H decomposition of G-rep, then $V_i^{(n)}$ appears for any $n \in N(H)$ as well!

Weyl groups

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$\alpha : H \longrightarrow H$ group automorphism

$\Rightarrow$ takes reps to reps: $\pi^\alpha(h) = \pi(\alpha(h))$

$\pi^\alpha(h_1 h_2) = \pi(\alpha(h_1 h_2)) = \pi(\alpha(h_1)\alpha(h_2))$

$= \pi^\alpha(h_1)\pi^\alpha(h_2)$.

$H \subset G \Rightarrow N(H) \subset G$ acts on H by automorphisms:

$\alpha_n(h) = n^{-1} h n =: h^{(n)}$ ... in fact $\underline{N(H)/Z(H)}$ acts

$G \circlearrowright V = \bigoplus V_i$ decomposition into H representations

$\Rightarrow$ for $v \in V_i$ $\quad nv \in (V_i)^\alpha$: $\quad n : V_i \longrightarrow (V_i)^\alpha$

$h \cdot nv = n h^{(n)} v$: set of nv $v \in V_i$ is again an H rep,

under the isom $n : V_i \longrightarrow nV_i$ h action taken to $h^{(n)}$ action.

$SU_2 \circlearrowright V$ finite dim irrep

$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} = U(1) = \mathbb{T} \subset SU_2$ $\quad V = \bigoplus_{i \in \mathbb{Z}} V_i$ $\quad U(1)$ reps.

$a^{-1} = \bar{a}$, $a \in U(1)$

$\Rightarrow$ look at $N(U(1))/U(1) = W$ Weyl group.

W generated by $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \in N(\mathbb{T})$: order 2 mod $U(1)$

__not__ order 2 in $SU(2)$!

$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix}$ $\quad$ ie $a \longrightarrow a^{-1}$

automorphism of $\mathbb{T}$.

sends weight $a \mapsto a^i$ to $a \mapsto a^{-i}$

so weights symmetric in __any rep__ of $SU(2)$.

$SU_n \circlearrowright V \Rightarrow V = \bigoplus_{\substack{i_1, \dots, i_n \in \mathbb{Z} \\ \sum i_j = 0}} V_{i_1, \dots, i_n}$ $\quad \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \mapsto a_1^{i_1} \cdots a_n^{i_n}$

$T = U(1)^{n-1} \subset SU_n$

$N(T) =$ permutation matrices $\begin{pmatrix} & x & \\ x & & \\ & & x \end{pmatrix}$

$N(T)/T = W = S_n$ symmetric group on n letters

acts on $\mathbb{Z}^{n-1}_{\sum i_j = 0} \subset \mathbb{Z}^n \Rightarrow$ symmetric patterns.

In our case: $N(T)$ acts, T acts <u>trivially</u> (abelian)
$\Rightarrow N(T)/T$ acts $= W$ Weyl group

$p: \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \in N(T)$ generates $W = \mathbb{Z}/2$ (note: <u>not</u> order two in $SL_2\mathbb{C}$)

What is this symmetry of $T = \mathbb{C}^*$?

$p^{-1} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ $\qquad \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix}$

ie $[a] \longrightarrow [a^{-1}]$ symmetry of $\mathbb{C}^*$.

$a^i \longmapsto a^{-i} \Rightarrow$ symmetry $i \longleftrightarrow -i$ of weight spaces.

<u>$SL_2\mathbb{C}$ vs $\mathfrak{sl}_2\mathbb{C}$ vs SU_2</u> can differentiate rep of $SL_2\mathbb{C}$ to $\mathfrak{sl}_2\mathbb{C}$. $\quad a = e^t \quad \frac{d}{dt}\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$\pi\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \cdot v = e^{nt} v$
$\Rightarrow \pi\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \cdot v = n v$

$SU_2 \subset SL_2\mathbb{C}$: simply connected Lie group, Lie algebra
$\mathfrak{su}_2 = \mathrm{span}(i, j, k) = \mathrm{Lie}\, SO_3$.

$$\mathfrak{su}_2 : \underset{i}{\begin{pmatrix} i & \\ & -i \end{pmatrix}} \quad \underset{j}{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}} \quad \underset{k}{\begin{pmatrix} & i \\ i & \end{pmatrix}} \quad = \text{pure vector quaternions}$$

$\mathfrak{su}_2 \otimes \mathbb{C} = \mathfrak{sl}_2\mathbb{C} = \mathfrak{sl}_2\mathbb{R} \otimes \mathbb{C}$: $\mathbb{C}$ linear span

- From rep theory POV, $\mathfrak{g}$ & $\mathfrak{g}\otimes\mathbb{C}$ the same! Acting in $\mathbb{C}$-<u>vector spaces</u> V, any $\mathbb{C}$ rep of $\mathfrak{g}$ extends automatically to $\mathfrak{g}\otimes\mathbb{C}$.
- $u \in \mathfrak{su}_2 \Rightarrow \exp(tu) = \cos t - u \sin t$
  $u^2 = -1$ as quaternion $\qquad$ 1-parameter group of rotations around u

So $\exp : \mathfrak{su}_2 \to SU_2$ surjective (true for any K compact)

$\Rightarrow$ directly exponentiate reps
$\mathfrak{su}_2 \leftrightarrow SU_2$

<u>Weyl's Unitary Trick</u> Tie in finite dimensional reps of
$$SL_n\mathbb{R} \subset SL_n\mathbb{C} \supset SU_n$$
$$G_{\mathbb{R}} \subset G_{\mathbb{C}} \supset K_{\mathbb{R}}$$

$$SL_n\mathbb{R} \circlearrowright V \Rightarrow \mathfrak{sl}_n\mathbb{R} \circlearrowright V \Rightarrow$$
$$\mathfrak{sl}_n\mathbb{C} \circlearrowright V \Rightarrow \mathfrak{su}_n \circlearrowright \mathbb{C} \Rightarrow SU_n \circlearrowright \mathbb{C}$$

Complete reducibility for $SL_n\mathbb{R}$: $W \subset V$ $SL_n\mathbb{R}$ invariant
$\Rightarrow$ $\mathfrak{sl}_n\mathbb{R} \to \mathfrak{sl}_n\mathbb{C} \to \mathfrak{su}_n \to SU_n$ invariant
$\Rightarrow$ $\exists$ SU_n invariant complement $\rightsquigarrow \mathfrak{sl}_n\mathbb{R} \rightsquigarrow SL_n\mathbb{R}$.

So we know all f.d. reps of these groups, algebras..

What about SO_3?

$SO_3 = SU_2/\pm 1$: reps of SO_3 = reps of SU_2 where -1 acts as 1.
e.g. $\mathbb{C}^2$ not rep of SO_3, $\mathbb{C}^3 = \mathbb{R}^3 \otimes \mathbb{C}$ is.
$\rightarrow$ need weights <u>even</u> $\quad 0 \; 1 \; -2 \; 0 \; 2 \; \cdots$
... only $2\mathbb{Z} \subset \mathbb{Z}$ of reps.

~~Characters~~ Construction

$\mathbb{C}^2 = \mathbb{C}\{x, y\}$ $\quad hx = x \quad hy = -y \quad ex = 0 \quad fx = y$..

$\mathrm{Sym}^2\mathbb{C}^2 = \mathbb{C}\{x^2, xy, y^2\}$

[$G \circlearrowright V \Rightarrow G \circlearrowright \mathrm{Sym}^n V$, $\mathfrak{g} \circlearrowright \mathrm{Sym}^n V$ as derivations]

$$\left.\frac{d}{dt}\right|_{t=0} (\gamma_t(v) \otimes \gamma_t(w)) = \gamma_0'(v) \otimes w + v \otimes \gamma_0'(w)$$

$$h(x^2) = x\,h(x) + h(x)\,x = 2x^2 \qquad h(y^2) = -2y^2$$
$$h(xy) = x\,h(y) + h(x)\,y = xy - xy = 0$$

$$\boxed{\text{Example: } V^{(2)} = \mathfrak{sl}_2 \text{ adjoint rep}}$$

<u>Realization of reps of $\mathfrak{sl}_2\mathbb{C}$</u>

<u>Symmetric power</u>: $V^{\otimes n} / (\ldots x \otimes y \ldots - \ldots y \otimes x \ldots) = I$

identify expressions $x \otimes y$ & $y \otimes x$ — write xy.

$\mathfrak{g}$ acts on $V \Rightarrow$ on $V^{\otimes n}$ by $f \cdot (v_1 \otimes v_2 \otimes \ldots v_n)$
$= fv_1 \otimes v_2 \otimes \ldots + v_1 \otimes fv_2 \otimes \ldots$

e.g. $f(v_1 \otimes v_2) = fv_1 \otimes v_2 + v_1 \otimes fv_2$
$= \frac{d}{dt}(e^{tf} v_1 \otimes e^{tf} v_2)|_{t=0}$] preserves I

$fg(xy) - gf(xy) = f((gx)y + x(gy)) - g((fx)y \ldots)$

So $\mathfrak{sl}_2\mathbb{C}$ acts on $\mathrm{Sym}^n \mathbb{C}^2 = \{y^n \; y^{n-1}x \ldots y^n\}$

$\mathbb{C}^2$: $e: y \to x$ $f: x \to y$ $h: x \to x, y \to -y$

Check: $e = x\frac{\partial}{\partial y}$ $f = y\frac{\partial}{\partial x}$ $h = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$

Let $z = \frac{x}{y}$. What does this mean? think of x, y as linear functionals on <u>dual</u> $\mathbb{C}^2$ take their ratio.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left(x\begin{pmatrix}\lambda \\ \mu\end{pmatrix}\right) = x\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\lambda \\ \mu\end{pmatrix} = \begin{pmatrix} a\lambda + b\mu \\ c\lambda + d\mu\end{pmatrix}\right) = a\lambda + b\mu$

$= (ax + by)\begin{pmatrix}\lambda \\ \mu\end{pmatrix}$ $\begin{pmatrix} a & b \\ c & d\end{pmatrix} y = cx + dy$.

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\frac{x}{y} = \frac{ax + by}{cx + dy} = \frac{y\left(a\frac{x}{y} + b\right)}{y\left(c\frac{x}{y} + d\right)} = \frac{az + b}{cz + d}$$

$\frac{\partial f}{\partial(\frac{x}{y})} = \cancel{\frac{df}{dx}}$ $\quad d\frac{x}{y} = \frac{y\,dx - x\,dy}{y^2}$

$\cancel{\frac{df}{d\frac{x}{y}} = y^2 \frac{df}{y\,dx - x\,dy}}$

$\frac{df}{dy} = \frac{df}{dz}\frac{dz}{dy} = \frac{df}{dz}\left(-\frac{x}{y^2}\right)$

$\frac{df}{dx} = \frac{df}{dz}\frac{dz}{dx} = \frac{df}{dz}\frac{1}{y}$ $\quad \boxed{y\frac{\partial}{\partial x} = \frac{\partial}{\partial z}}$ $\quad x\frac{df}{dy} = -z^2\frac{df}{dz}$ $\quad x\frac{\partial}{\partial y} = -z^2\frac{\partial}{\partial z}$

SL_2
B-W 2

Recall: holomorphic vector fields on $\mathbb{CP}^1$ =
$$\left\{ \underset{e}{\frac{\partial}{\partial z}},\ \underset{h}{-2z\frac{\partial}{\partial z}},\ \underset{f}{-z^2\frac{\partial}{\partial z}} \right\}$$

Also note: $SL_2\mathbb{C}$ acts on $\mathbb{C}^2$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} aw_1 + bw_2 \\ cw_1 + dw_2 \end{pmatrix} \Rightarrow \text{ up to scalar:}$$
act on $\mathbb{CP}^1$ $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \sim \begin{pmatrix} z = \frac{w_1}{w_2} \\ 1 \end{pmatrix}$ for $w_2 \neq 0$

$z \mapsto \frac{az+b}{cz+d}$. $z = \infty \Leftrightarrow w_2 = 0 \Leftrightarrow \begin{pmatrix} * \\ 0 \end{pmatrix}$

Stabilizer of $z = \infty \Leftrightarrow$ of line $\begin{pmatrix} * \\ 0 \end{pmatrix}$ $= B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ Borel

(stabilizer of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ nilpotent)

$\Rightarrow \mathbb{P}^1 = SL_2\mathbb{C}/B$.

More geometry: $T = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ acts by rotation $\Rightarrow -2z\frac{\partial}{\partial z}$

$N = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ acts by translation $\xrightarrow{\text{differentiate}} \frac{\partial}{\partial z}$

$B_- = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} = \mathrm{Stab}\ 0$ acts by $z \mapsto \frac{az}{cz + a^{-1}}$

$\mathbb{P}^1 = SU_2/T = S^2$: $SU_n \cap B \subset T$ so $SU_n/T = G/B$

involution $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \in SL_2$ conjugates $B \to B_-$, acts as $z \mapsto \frac{-1}{z}$. On T acts as $a \mapsto a^{-1}$

No holomorphic functions on $\mathbb{P}^1$ ... but do have "homogeneous coords" $x, y \in (\mathbb{C}^2)^*$.

What are these on $\mathbb{P}^1$: $f(\lambda v) = \lambda f(v)$
in particular ratios
$\frac{f_1}{f_2}(\lambda v) = \frac{f_1}{f_2}(v)$ are functions! e.g. $\frac{x}{y} = z$, but have poles

SL_2
B-W 3

On $\mathbb{P}^1$ have complex line bundle $\mathcal{T}$:
each point in $\mathbb{P}^1 \longrightarrow$ line in $\mathbb{C}^2$: $\mathrm{Span}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \mathcal{T}_z$

x, y are linear functions on $\mathbb{C}^2 \Rightarrow$ for each $z \in \mathbb{P}^1$
they give linear functions of $\mathcal{T}_z \longrightarrow$ elements
of dual line $\mathcal{T}_z^*$ ... line bundle $\underline{\mathcal{O}(1)}$

$V^* = \mathbb{C}\{x,y\} \twoheadrightarrow V/\mathrm{ann}(\mathcal{T}_z) = \mathcal{T}_z^*$

Homogeneous polys of deg k = $\mathrm{Sym}^k\, \mathbb{C}\{x,y\}$:
again ratio is constant on line $f(\lambda v) = \lambda^k f(v)$
... element of $\mathrm{Sym}^k \mathcal{T}_z^*$
$\Rightarrow \mathrm{Sym}^k \mathcal{O}(1) = \mathcal{O}(k)$, global sections $= \mathrm{Sym}^k V^*$.

~~SL_2 acts on~~

Alternatively: $\bigoplus_k \mathrm{Sym}^k V^* = \mathbb{C}[\mathbb{C}^2]$ all polynomial functions on $\mathbb{C}^2$, we're decomposing it under action of $\mathbb{C}^* = \begin{pmatrix} a & \\ & a \end{pmatrix}$, commutes with $SL_2\mathbb{C}$.
eigenspaces = $SL_2\mathbb{C}$ irreps.

$$\mathbb{C}^2 \setminus 0 = SL_2 / \left(N = \mathrm{Stab}\begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$\mathbb{C}[SL_2/N] = \mathbb{C}[\mathbb{C}^2] = \bigoplus_{V \text{ irrep}} V^*$$

From $\mathbb{P}^1$ point of view: $\bigoplus \Gamma(\mathcal{O}(k)) = \bigoplus_{V \text{ irrep}} V^*$

Calculate: $\begin{pmatrix} & 1 \\ & \end{pmatrix} \cdot x(v) = \frac{d}{dt} e^{t\begin{pmatrix} & 1 \\ & \end{pmatrix}} \cdot x(v) \Big|_{t=0}$

$$= \frac{d}{dt}\Big|_{t=0} x\left(\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = \frac{d}{dt}\Big|_{t=0} (\lambda + t\mu) = \mu = y(v)$$

like $\begin{pmatrix} & 1 \\ & \end{pmatrix} \cdot y(v) = 0 \Rightarrow$

$$\begin{pmatrix} & 1 \\ & \end{pmatrix} \cdot x^m y^n \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = D\, x^m y^n \begin{pmatrix} \lambda + t\mu \\ \mu \end{pmatrix} = D\, \mu^n (\lambda + t\mu)^m$$

$$= m x^{m-1} y^{n+1} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \qquad \begin{pmatrix} & 1 \\ & \end{pmatrix} \longleftrightarrow y \frac{\partial}{\partial x}$$

Another geometric POV:

$PSL_2\mathbb{C} \circlearrowright Sym^n V^*$ action on irrep

Preserves vectors of form $[v^n]$ $v \in V^*$

$\mathbb{P}^1 = \mathbb{P}V^* \hookrightarrow \mathbb{P}^n = \mathbb{P}(Sym^n V^*)$

Veronese embedding $[v] \longmapsto [v^n]$

In bases $[x, y] \longmapsto [x^n, x^{n-1}y, x^{n-2}y^2 \ldots y^n]$

From another POV: $v \in W$ highest weight vector:

$ev = 0$, $hv = \lambda v$ $\Rightarrow$ $[v]$ is fixed by B

$\Rightarrow$ orbit Φ $SL_2\mathbb{C} \cdot [v] \twoheadleftarrow \mathbb{P}^1$ $\Rightarrow$ orbit is $\mathbb{P}^1$

(no coverings from $\mathbb{P}^1$).

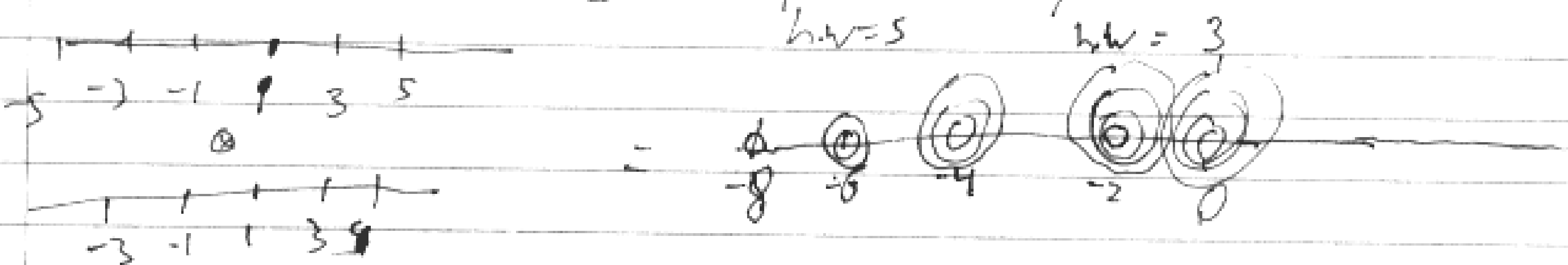
W irreducible $\Rightarrow$ Φ spans W - not contained in any proper subspace! - rational normal curve

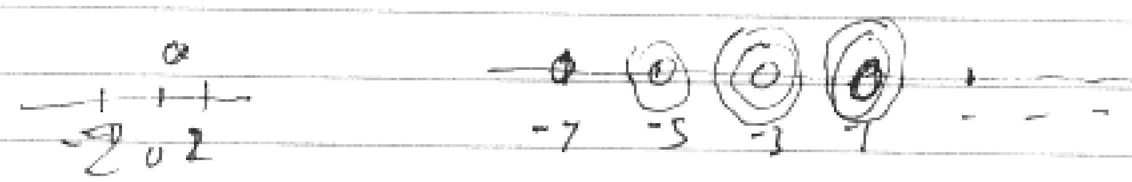
$t \mapsto (t, t^2, t^3)$

Every $[v'] \in \Phi$ fixed by a Borel subgroup B' conjugate to B --- possible highest weight vectors.

In fact for V irred $\exists!$ closed $SL_2\mathbb{C}$ orbit $\Phi \subset W$ ..... no other compact homog spaces, & only 1-dim space of B-invariants....

Decomposing reps e.g. $Sym^5 V \otimes Sym^3 V$





$Sym^7 \oplus Sym^5 \oplus Sym^3$

Note have canonical ~~proj~~ subrep $Sym^{m+n} \subset Sym^m \otimes Sym^n$

$\Rightarrow \bigoplus Sym^i$ form ring.