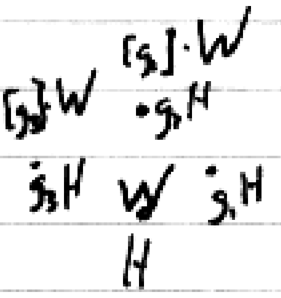
Induced Representations

Finite $G \supset H$ W H rep $\Rightarrow \mathrm{Ind}_H^G W = V$ is a G-rep with property $V = \bigoplus_{[g]\in G/H} [g]\cdot W$

Definition $V = \mathrm{Map}_H(G, W) = \{ f : G \to W$ s.t. $f(gh^{-1}) = h\, f(g) \}$ G acts by left transl.



Note value of f on each H-coset determined by definition, cosets don't interact $\Rightarrow$ get characteristic property.

Get vector bundle over G/H with fibers $\sim W$.

$\Rightarrow$ consider
$$\begin{tikzcd} G \arrow[d, "H"] \\ G/H \end{tikzcd}$$
H-bundle. Take associated vector bundle

$$\begin{tikzcd} G\times_H W = G\times W/(g,w)\sim(gh^{-1}, hw) \arrow[d] \\ G/H \end{tikzcd}$$
. Fibers all vector spaces $\cong W$.

This is a G-equivariant vector bundle:

For every $g \in G$ get isomorphism $G\times_H W|_{g_1} \xrightarrow[g]{\sim} G\times_H W|_{g\cdot g_1}$

$\mathrm{Map}_H(G, W)$ = sections of $G\times_H W$:

$g \in G \longleftrightarrow f(g) \in W$

$f(gh^{-1}) = h\cdot f(g)$ means descends to G/H.

Examples:

G, H Lie groups: define
$$\begin{tikzcd} G\times_H W \arrow[d] \\ G/H \end{tikzcd}$$
the same way, now have lots of different notions of sections

$\Leftrightarrow$ $C^\infty_H(G, W)$, $C^\omega_H(G, W)$, $L^2_H(G, W)$ (careful)

or holomorphic/polynomial $\mathbb{C}_H[G, W]$ if G/H holomorphic...

Our example: $T = \begin{pmatrix} q & \\ & q^{-1} \end{pmatrix} \subset B = \begin{pmatrix} q & b \\ & q^{-1} \end{pmatrix} \subset SL_2\mathbb{C}$

$T \circlearrowleft \mathbb{C}_n \; : \; \begin{pmatrix} q & \\ & q^{-1} \end{pmatrix} \longmapsto q^n.$

Expand to $B \longrightarrow B/N \simeq T \longrightarrow \mathbb{C}^*$
$b = \begin{pmatrix} q & b \\ & q^{-1} \end{pmatrix} \longmapsto q^n = \chi_n(b)$

Now take holomorphic induction

$$\mathrm{Ind}_B^G \mathbb{C}_n = \{ f \in \mathbb{C}[SL_2] \text{ holomorphic}: f(gb^{-1}) = \chi_n(b)\, f(g) \}$$

Such f are automatically N-invariant: $\overset{\chi_{-n}''(b)}{} \; f \in \mathbb{C}[\underset{\mathbb{C}^2-0}{\underset{\shortparallel}{SL_2/N}}] \circlearrowleft H$

prescribe action of $\mathbb{T}$: $\mathbb{T}$ acts by rescaling

<u>Careful</u>: T acts with weight one on line $\ell \in \mathbb{P}^1$
$\Rightarrow$ weight -1 on line ℓ^* of homogeneous polynomials

$\Rightarrow \quad \mathrm{Ind}_B^G \mathbb{C}_{-n} = V^{(n)}.$

Geometrically $G \times_B \mathbb{C}_n = \mathcal{O}(-n)$ $\qquad G \times_B \mathbb{C}_1 =$ tautological line

$$\Gamma(G \times_B \mathbb{C}_{-n}) = \begin{cases} 0 & n < 0 \\ V^{(n)} & n \geq 0 \end{cases}$$

[$SL_n\mathbb{C}$: $\quad \chi : \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix} \longrightarrow \mathbb{C}^* \ni q_1^{l_1} \cdots q_n^{l_n}$

$\chi \in T^\vee \simeq \mathbb{Z}^{n-1} \qquad B \longrightarrow B/N \simeq T \longrightarrow \mathbb{C}^*$, with $B \xrightarrow{\chi} \mathbb{C}^*$

$\mathrm{Ind}_B^G \mathbb{C}_\chi = \Gamma(G \times_B \mathbb{C}_\chi)$ always irreducible,
all irreps arise]

[$\mathbb{P}^1 = SU(2)/U(1) \Rightarrow$ can take $SU(2) \times_{U(1)} \mathbb{C}_n$ same line bundle. can take all continuous sections etc — don't know it's holomorphic!]

Group algebra induction: (G finite) $\mathbb{C}G$ algebra $\supset \mathbb{C}H$

Can take tensor products of $\mathbb{C}H$ modules:

$$\mathrm{Ind}_H^G W = \mathbb{C}G \underset{\mathbb{C}H}{\otimes} W \qquad (a \otimes rb = ar \otimes b)$$

$$A \underset{R}{\otimes} B$$

... again putting in condition [illegible]

$$(g, w) \sim (gh^{-1}, hw) \Leftrightarrow (g, hw) \sim (gh, w)$$

Advantage of algebra: $R_1 \longrightarrow R_2$ ring homomorphism

A R_1-mod
B R_2-mod
$\Rightarrow [B]_{R_1}$ R_1-mod

$$\boxed{\mathrm{Hom}_{R_1}(A, [B]_{R_1}) = \mathrm{Hom}_{R_2}(R_2 \underset{R_1}{\otimes} A, B)}$$

$\longrightarrow$: generate from $1 \otimes A$ & R_2 action on B

$\longleftarrow$: restrict to $A \longrightarrow R_2 \underset{R_1}{\otimes} A \longrightarrow B$.

$\Rightarrow$ Frobenius reciprocity

$$\mathrm{Hom}_H(W, \mathrm{Res}_G^H V) = \mathrm{Hom}_G(\mathrm{Ind}_H^G W, V)$$

$$\langle W, \mathrm{Res}\, V \rangle_H = \langle \mathrm{Ind}\, W, V \rangle_G$$

adjoint "operators"

Lie algebra analog: ~~E~~ Enveloping Algebras

Analogy: G is to $\mathbb{C}G$ as $\mathfrak{g}$ is to $U\mathfrak{g}$

$$(G : \mathbb{C}G :: \mathfrak{g} : U\mathfrak{g})$$

i.e in any G rep can find more operators acting: take linear combos of operators from G.

in any $\mathfrak{g}$ rep can find more operators: take powers of operators coming from $\mathfrak{g}$

So to $G, \mathfrak{g}$ both assign associative algebras [in fact Hopf algebras] $\mathbb{C}G$, $U\mathfrak{g}$.

$U\mathfrak{g}$ = quotient $\overset{\otimes^{\bullet}}{\cancel{T}^{\bullet}}\mathfrak{g} / \{ \underset{\mathfrak{g}^{\otimes 2}}{\underset{\uparrow}{xy-yx}} = \underset{\mathfrak{g}}{\underset{\uparrow}{[x,y]}} \}$ of tensor algebra on $\mathfrak{g}$ by ideal

make $xy-yx$ act as it should, as $[x,y]$

e.g $\mathfrak{g}$ abelian $\Rightarrow$ $U\mathfrak{g} = \otimes^{\bullet}\mathfrak{g} / xy-yx=0 = \mathrm{Sym}^{\bullet}\mathfrak{g}$

This relation is satisfied by operators in any rep of $\mathfrak{g}$ $\Rightarrow$ wherever $\mathfrak{g}$ acts, so does $U\mathfrak{g}$.

$\mathfrak{g}$-reps $\Longleftrightarrow$ $U\mathfrak{g}$-reps, (associative unital algebra)

$U\mathfrak{sl}_2$:

$eh \quad ef \quad hf \quad e^2 \quad f^2 \quad h^2$ deg 2 : PBW thm: basis given by ordered monomials in e, f, h.

$e \quad h \quad f$ deg 1 :

1 deg 0

Pf: Surely these things span.

The ideal $\langle ef-fe-h \rangle$ doesn't contain any ordered polynomials. More generally, $\mathrm{Sym}^{\bullet}\mathfrak{g} = \mathrm{Gr}(U\mathfrak{g})$.

$U\mathfrak{g}$ geometrically: $\mathfrak{g}$ comes from vector fields on some manifold X $\Rightarrow$ $U\mathfrak{g}$ comes from multiplying those — i.e. differential operators

$\mathfrak{g} \simeq$ left invariant ~~d~~ vector fields on G

$U\mathfrak{g} \simeq$ " " diff ops on G

e.g $\mathfrak{g} = \mathfrak{sl}_2$ acts on $\mathbb{P}^1$ via $e \mapsto \frac{\partial}{\partial z}$, $h \mapsto -2z\frac{\partial}{\partial z}$

$f \mapsto -z^2 \frac{\partial}{\partial z}$

$\Rightarrow$ get operators like $e^2 \mapsto \frac{\partial^2}{\partial z^2}$

$fe^2 \mapsto -z^2 \frac{\partial^3}{\partial z^3}$ etc..

Can now define Lie algebra induction:

Let $\mathbb{C}_\lambda$ 1-dim vector space spanned by vector v_λ, with action of $\mathfrak{b} = \mathrm{Span}\{e, h\} \subset \mathfrak{g}$ Borel subalgebra

$e \cdot v_\lambda = 0 \quad h \cdot v_\lambda = \lambda v_\lambda$.

Note $\lambda \in \mathbb{C}$ can be anything, unlike with group version $\mathbb{C}_n$ $n \in \mathbb{Z}$.

Now set $V_\lambda := U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda$ tensor product over $U\mathfrak{b}$: ie $ar \otimes b \sim a \otimes rb$

or explicitly, need only specify

$$ah \otimes v_\lambda = a \otimes hv_\lambda = \lambda\, a \otimes v_\lambda$$
$$ae \otimes v_\lambda = a \otimes ev_\lambda = 0.$$

$\Rightarrow$ as a vector space V_λ looks like polynomials in f (since we already prescribed e, h actions)

$$\dots\ f^4 v_\lambda \quad f^3 v_\lambda \quad f^2 v_\lambda \quad f v_\lambda \quad v_\lambda$$

V_λ is an infinite dimensional $U\mathfrak{g}$, hence $\mathfrak{g}$, module.

... a $\boxed{\text{Verma module}}$

Recall we proved a relation

$$e f^{n+1} v = f^{n+1} e v + (n+1) f^n (h-n) v$$

for $v \in V$ any rep of $\mathfrak{g}$ ... in fact, just can say

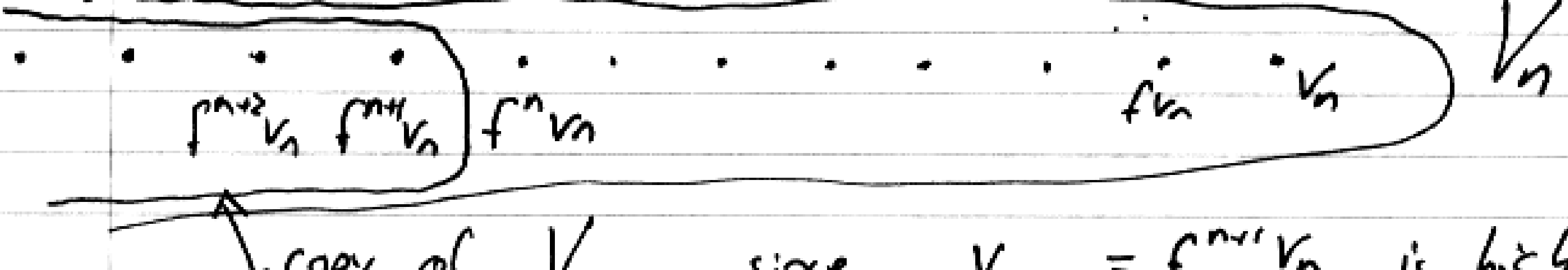
$$\boxed{e f^{n+1} = f^{n+1} e + (n+1) f^n (h-n)} \text{ in } U\mathfrak{g}. \quad \mathfrak{g} = \mathfrak{sl}_2$$

So now look at $V_\lambda = V_n$ for $\lambda = n$ positive integer.

In V_n we have a nonzero vector $f^{n+1} v_n$.

But $$e \cdot (f^{n+1} v_n) = f^{n+1} (\underset{\overset{\shortparallel}{0}}{e \cdot v_n}) + (n+1) f^n \cdot (\underset{\overset{\shortparallel}{0}}{(h-n) v_n})$$
$$= 0$$

So rep V_n has a subrepresentation



copy of V_{-n-2} since $v_{-n-2} = f^{n+1} v_n$ is highest weight vector of weight $-n-2$ ...

The quotient V_n / V_{-n-2} is just our favorite fin dim rep $V^{(n)} = \mathrm{Sym}^n \mathbb{C}^2$.

So have $0 \longrightarrow V_{-n-2} \underset{\text{injective}}{\longrightarrow} V_n \underset{\text{surjective}}{\longrightarrow} V^{(n)} \longrightarrow 0$

but not split:

$V_n \not\cong V^{(n)} \oplus V_{-n-2}$ since V_n indecomposable: can get from v_n to any other vector by applying f's ...

... our first example of lack of complete reducibility for sl_2 ....

The Casimir. Another key reason to introduce $U\mathfrak{g}$: it has interesting operators in it which aren't in $\mathfrak{g}$ itself.

Main example: Casimir: take an inner product $\langle , \rangle$ on $\mathfrak{g}$ which is invariant under G acting by conjugation $\langle gx, gy \rangle = \langle x, y \rangle$, let e_i be a basis for $\mathfrak{g}$ & e^i the dual basis wrt $\langle , \rangle$

$\Rightarrow$ let $C = \sum e_i \cdot e^i \in U\mathfrak{g}$

from general principles [... independent of choice of basis, in fact invariant under G ... : let's see this explicitly for sl_2:

take $\langle x, y \rangle = \mathrm{Tr}(xy)$ $\quad x, y \in \mathfrak{sl}_2\mathbb{C}$ matrices

$\langle e, f \rangle = \mathrm{Tr}\begin{pmatrix} & 1 \\ & \end{pmatrix}\begin{pmatrix} & \\ 1 & \end{pmatrix} = 1$, $\quad \langle h, h \rangle = \mathrm{Tr}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}^2 = 2$

So $C = ef + fe + \frac{1}{2} h^2$

Invariance: C commutes with $\mathfrak{sl}_2$:

$Ce = eC \qquad Cf = fC \qquad Ch = hC$

~~[illegible]~~

e.g can check [IF YOU WANT] explicitly

use
$ef = fe + h$
$fe = ef - h$
$he = eh + 2e$
etc
commutation
relation

$$
\begin{aligned}
Ce &= efe + fee + \tfrac{1}{2} h^2 e \\
&= (eef - eh) + (efe - he) + \left(\tfrac{1}{2} heh + he\right) \\
&= eef + efe \quad - eh + \left(\tfrac{1}{2} eh^2 + eh\right) \\
&= eef + efe + \tfrac{1}{2} eh^2 = eC .
\end{aligned}
$$

Since $\mathfrak{g}$ generates $U\mathfrak{g}$, it follows that C commutes with all of $U\mathfrak{g}$, ie is in the centre of $U\mathfrak{g}$, denoted $Z\mathfrak{g}$.

~~SKIP FIRST TO GEOM INTER~~

Schur's lemma $\Rightarrow$ C (being an element of $Z\mathfrak{g}$, hence commuting with all operators from $\mathfrak{g}$) must act as scalar multiplication on any irrep of $\mathfrak{g}$.

So as a way to decompose reps of $\mathfrak{g}$ can break up into eigenspaces of C.

Check eg on $V^{(n)}$: $\quad e \cdot v_n = 0 \quad h \cdot v_n = n v_n$

$$
\begin{aligned}
\Rightarrow \quad C \cdot v_n &= ef\, v_n + f(\overset{0}{\overset{\shortparallel}{e v_n}}) + \tfrac{1}{2} h^2 v_n \\
&= (\overset{\shortparallel}{fe\, v_n} + h v_n) + \tfrac{1}{2} n^2 v_n \\
&= n v_n + \tfrac{1}{2} n^2 v_n = n\left(\tfrac{n}{2} + 1\right) v_n
\end{aligned}
$$

$\Rightarrow$ C acts as $n(\frac{n}{2}+1)$ on $V^{(n)}$

Note this is same for n and $-n-2$!

~~Harish-Chandra isomorphism for sl2~~ !

~~$Z sl_2 = \mathbb{C}[C] \cong \mathbb{C}[z]^{\mathbb{Z}/2}$~~

~~ie nothing in center but powers of C~~

~~Polynomials in one variable~~

[Can skip this page]

In fact same argument works for Verma V_λ for any $\lambda \in \mathbb{C}$: C acts by scalar $\lambda(\frac{1}{2}\lambda+1)$ even though V_λ not always irreducible – it's indecomposable, generated by v_λ, so C still acts by same scalar on all V_λ as it does on v_λ (ie $C f^n v_\lambda = f^n C v_\lambda$ $= \lambda(\frac{1}{2}\lambda+1) \cdot f^n v_\lambda$)

$\Rightarrow$ C gives function $\lambda(\frac{1}{2}\lambda+1)$ on λ-line. This function generates $\mathbb{C}[\lambda]^{\mathbb{Z}/2}$, polynomials in λ invariant under $\mathbb{Z}/2$ action taking $\lambda \mapsto -\lambda-2$.

Harish-Chandra isomorphism ($\mathfrak{sl}_2$)

$$\begin{array}{ccc} \underbrace{Z\mathfrak{sl}_2 = \mathbb{C}[C]}_{\text{center is just powers of } C} & \simeq & \mathbb{C}[\lambda]^{\mathbb{Z}/2} \\ & & \| \\ & & \mathbb{C}[\mathfrak{h}^*]^W : \end{array}$$

λ is possible eigenvalue of h, i.e. λ (weight) is a linear functional from $\mathfrak{h} = \mathbb{C}\cdot h \longrightarrow \mathbb{C}$, $h \longmapsto \lambda$

$W = \mathbb{Z}/2$ is the Weyl group,

but acting by $\lambda \mapsto -\lambda-2$, ie reflection in the point $\lambda = -1$.

GEOMETRIC INTERP of Casimir

Recall $U\mathfrak{g} \simeq$ left invariant diffops on G

What is $Z\mathfrak{g}$? $\mathfrak{g}$ action on $U\mathfrak{g}$ comes from right action of G on G, by differentiation.
Center means diffops commuting with this action as well $\Rightarrow$ $\boxed{Z\mathfrak{g} = \text{bi-invariant diffops on } G.}$

Casimir: canonical quadratic bi-invariant operator
— Laplacian, on G. also
descends to G/H any homogeneous space,
since it is invariant $\Rightarrow$ Laplacian on homogeneous spaces.

Try on $\mathbb{P}^1$: $e \mapsto \frac{\partial}{\partial z}$, $h \mapsto -2z\frac{\partial}{\partial z}$, $f \mapsto -z^2\frac{\partial}{\partial z}$

$fe \mapsto -z^2 \frac{\partial^2}{\partial z^2}$ $\quad ef \mapsto \frac{\partial}{\partial z}\left(-z^2\frac{\partial}{\partial z}\right) = -2z\frac{\partial}{\partial z} - z^2\frac{\partial^2}{\partial z^2}$

$\frac{1}{2}h^2 \mapsto \frac{1}{2}\left(-2z\frac{\partial}{\partial z}\right)\left(-2z\frac{\partial}{\partial z}\right) = 2z^2\frac{\partial^2}{\partial z^2} + 2z\frac{\partial}{\partial z}$

So $C = ef + fe + \frac{1}{2}h^2 \longmapsto 0$ exactly cancel!

Not surprising! holomorphic functions on $\mathbb{P}^1$ $\mathbb{C}[\mathbb{P}^1] = V^{(0)} = \mathbb{C}$
form the stupid 1-dim rep of $\mathfrak{sl}_2$
with $\lambda = 0 \Rightarrow C$ acts by 0
$\Leftrightarrow$ holomorphic functions killed by Laplacian!

In fact all diffops (holomorphic) on $\mathbb{P}^1$ given this way

$$\mathcal{D}(\mathbb{P}^1) = U\mathfrak{sl}_2 / (C - 0)U\mathfrak{sl}_2$$

[: quotient $U\mathfrak{sl}_2$ by ~~center~~ (2-sided) ideal gen by C

[$\mathcal{D}(G/B) = U\mathfrak{g} / (Z\mathfrak{g}$ acts like it does on trivial representation)
in general for flag varieties — Beilinson-Bernstein]

Characters & Peter-Weyl (mimics Ch. 9 in Segal's lectures on Lie groups)

G abelian group, (V,χ) irrep $\Rightarrow$ $V \simeq \mathbb{C}$

- think of (V,χ) as given by the function
  $g \longmapsto \operatorname{tr} \chi(g)$, which is just $\chi(g) \in \mathbb{C}^*$
  if we pick a basis $V \simeq \mathbb{C}$ & χ is a 1×1 matrix.

G nonabelian, (V,π) finite dimensional representation
$\Rightarrow$ can still assign function χ_π on G
via $\chi_\pi(g) = \operatorname{tr} \pi(g)$ trace. — Character of $(V,\pi$

$$\chi_\pi(h^{-1}gh) = \operatorname{tr} \pi(h^{-1}gh) = \operatorname{tr} \pi(g) = \chi_\pi(g).$$

class function (conjugation invariant).
If G is finite, can describe χ_π by giving its values on conjugacy classes in G.

e.g. $\chi_\pi(1) = \dim V$.

- More general construction: matrix elements, aka representative functions on G:
  aka algebraic functions

$v \in V$ (fin. dim) G rep, $v^* \in V^*$ dual space

$$\Rightarrow \quad f_{v,v^*}(g) = \langle v^*, g\cdot v\rangle \in \mathbb{C}$$

If V is a continuous rep $\Rightarrow$ f_{v,v^*} is continuous
holomorphic, smooth, ... $\Rightarrow$ ... holomorphic, smooth
(essentially by definition).

Thus we get a map $V \otimes V^* \longrightarrow C(G)$ continuous $\mathbb{C}$-functions, for example
$v \otimes v^* \longmapsto f_{v,v^*}$
(extend by linearity)

$V \otimes V^*$ is actually a rep of $\underset{\text{"left"}}{G} \times \underset{\text{"right"}}{G}$:

V rep of G_{right} , V^* rep of G_{left} (via $\langle h \cdot v^*, v \rangle := \langle v^*, h^{-1} v \rangle$)

So $g_1 \times g_2 \cdot v_1 \otimes v_2^* = g_1 \cdot v_1 \otimes g_2 \cdot v_2^*$

So is $C(G)$: $h_1 \times h_2 \cdot f(g) = f(h_1^{-1} g h_2)$
(ie $G \times G \curvearrowright G$)

Check : $(h_1 \times h_2) f_{v,v^*}(g) = \langle v^*, h_1^{-1} g h_2 \cdot v \rangle$
$= \langle h_1 v^*, g \cdot (h_2 v) \rangle = f_{h_2 v, h_1 v^*}(g)$

So $V \otimes V^* \longrightarrow C(G)$ as $G \times G$ reps.

Note : V <u>irreducible</u> $\Rightarrow$ this map is injective
since $V \otimes V^*$ is irreducible $G \times G$ rep
[$\cdot \operatorname{End}_{G \times G}(V \otimes V^*) = \operatorname{End}(V \otimes V^*)^{G \times G} = (\operatorname{End} V)^G \otimes \operatorname{End}(V^*)^G$
$\cong \mathbb{C}$ ...... enough if
G reps completely reducible - e.g G <u>compact</u>, finite,
or reductive e.g $SL_2 \mathbb{C}$ ...]

So $\mathbb{C}^{alg}(G) := \bigoplus_{\substack{V \text{ irrep} \\ \text{of } G, \\ \text{fin dim}}} V \otimes V^* \subset C(G)$

<u>**Characters**</u> : $V \otimes V^* = \operatorname{End} V \in \mathrm{Id}$ canonical element
$\mathrm{Id} =\cdot \sum e_i \otimes e_i^*$ in any basis e_i of V

V irreducible - this is the <u>unique</u> vector in $V \otimes V^*$
which is invariant under G acting diagonally
$G \xrightarrow[\text{diag}]{} G \times G$ (Schur's lemma:
$(V \otimes V^*)^G = (\operatorname{End} V)^G = \mathbb{C} \cdot \mathrm{Id}$)

$f_{\mathrm{Id}} = \sum f_{e_i, e_i^*} = \chi_V$ the character of V

... $\sum \langle e_i^*, \pi(g) e_i \rangle = \operatorname{tr}_V \pi(g)$.

So characters are matrix elements.

More on matrix elements : they form a ring:

def f_{V,v,v^*} = the matrix elt associated to: $v \in V$, $v^* \in V^*$

$\Rightarrow$ Exercise $f_{V,v,v^*} + f_{W,w,w^*} = f_{V\oplus W,\, v\oplus w,\, v^*\oplus w^*}$

$$f_{V,v,v^*} \cdot f_{W,w,w^*} = f_{V\otimes W,\, v\otimes w,\, v^*\otimes w^*}$$

$\Rightarrow$ get a ring $\mathbb{C}^{alg}[G]$ of representative functions / algebraic fns on G, $\mathbb{C}^{alg}[G] \subset C(G)$.

e.g. • $\mathbb{C}^{alg}[U(1)] = \bigoplus_{n\in\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}[z,z^{-1}] = \mathbb{C}[\mathbb{C}^*]$

polynomial functions on $\mathbb{C}^*$ = poly functions on $U(1)$ (finite Fourier series) = representative functions.

• $\mathbb{C}^{alg}[SU_2] = \bigoplus_{n\geq 0} \mathbb{C}^n \otimes \mathbb{C}^n$. As ring, generated by $V = \mathbb{C}^2$ defining representation: $\underset{\overset{\shortparallel}{\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right\}}}{\mathbb{C}^2 \otimes \mathbb{C}^2} \subset C(SU_2)$

Exercise (hint?) not independent: have relation $ad - bc = 1$ in $C(SU_2)$

$\Rightarrow \mathbb{C}^{alg}[SU_2] \simeq \mathbb{C}[SL_2\mathbb{C}] = \mathbb{C}[a,b,c,d]/ad-bc=1$

Orthogonality relations G compact (e.g. $SU(n)$),

$\int_G dg = 1$ normalized Haar measure

$\mathbb{C}^{alg}[G] \subset C(G) \subset L^2(G)$, can take $\langle f,g\rangle_{L^2} = \int_G f\bar{g}\,dg$

• Note: V unitary rep of G $\Rightarrow$

$V^* \simeq \bar{V}$ complex conjugate rep: i.e. same underlying vector space, complex conjugated action.

Proposition $\langle f_{V,v,v^*}, f_{W,w,w^*} \rangle = 0$ unless $V \simeq W$.

Reinterpret Schur's lemma: V irrep of group $G \Rightarrow$ ∃ at most one hermitian inner product on V preserved by G, up to scalar
— must be nondegenerate, or else its kernel
$\ker (,) : V \longrightarrow \bar{V}^*$ defines an invariant subspace.
& then $(,) : V \longrightarrow \bar{V}^*$ must be an isomorphism
$\Rightarrow$ unique up to scalar by Schur.

Similarly V, W nonisomorphic irreps $\Rightarrow$ any invariant inner product is $\oplus$ of one on V & one on W
(all nondiagonal components must vanish)

$\Rightarrow$ proposition.

Corollary Representations are determined by their character: characters of different reps are orthogonal.

In fact orthonormal:

Prop [see Segal lectures on reserve] ... easy calculation

$$\langle f_{V,v_1,v_1^*}, f_{V,v_2,v_2^*} \rangle_{L^2} = \frac{1}{\dim V} (v_1, v_2) \overline{(v_1^*, v_2^*)}$$

(here $(,)$ is any nondeg invariant inner product on V: ! up to scalar $\Rightarrow$ above well defined if where we define $( )$ on V^* correctly).

Corollary characters orthonormal basis for $\mathbb{C}^{\text{alg}}[G] \cap \mathcal{L}^2(G)^G$

where $\mathcal{L}^2(G)^G$ = ~~continuous~~ class functions: invariant under diagonal G action $f(g) \mapsto f(h^{-1}gh)$.

Peter-Weyl Theorem What is the role of $\mathbb{C}^{alg}[G]$?

One answer (algebraic geometry): there is naturally a complex Lie group, in fact complex algebraic group, $G_{\mathbb{C}}$ = the complexification of G.
defined in alg-geom speak as $G_{\mathbb{C}} = \mathrm{Spec}\, \mathbb{C}^{alg}[G]$

ie polynomial functions on $G_{\mathbb{C}} = \bigoplus_{\substack{V \text{ irrep} \\ \text{of } G_{\mathbb{C}}}} V \otimes V^*$

(recall reps of $G \leftrightarrow G_{\mathbb{C}}$!)

e.g. $G = SU_n \quad G_{\mathbb{C}} = SL_n\mathbb{C}, \quad G = SO(n,\mathbb{R}) \quad G_{\mathbb{C}} = SO(n,\mathbb{C})$

--- this gives a way to define $G_{\mathbb{C}}$ from G
(need also that $\mathbb{C}^{alg}[G]$ is a Hopf algebra ..... not for this class.]

Another answer! $\mathbb{C}^{alg}[G]$ plays the role of e^{inx} in Fourier series $\Longrightarrow$ analog for G:

Theorem (G compact) **I.** $\mathbb{C}^{alg}[G] \subset L^2(G)$ dense subspace,
i.e $L^2(G) = \overline{\bigoplus V \otimes V^*}$ orthonormal direct sum

$\Longleftrightarrow$ **II.** Any ~~unit~~ rep W of G is a completed direct sum of its V-isotypic components for V f.d. irrep of G

[V isotypic component $[W]_V$ = sum of all copies of V contained in W ... more formally

$$[W]_V = \mathrm{Im}\,\big(V \otimes \mathrm{Hom}_G(V, W) \longrightarrow W \big) \quad v \otimes f \longmapsto f(v) \;]$$

So $W = \widehat{\bigoplus_V [W]_V}$: ie $\bigoplus_V [W]_V$ is dense in W.

P-W 6

Discussion Key notion : G-finite vectors

V rep of $G \Rightarrow V^{fin} = \{ v \in V : v$ is contained in some fin. dim G-invariant subspace$\}$

$\Leftrightarrow$ Span$\{g.v : g \in G\}$ is finite dimensional.

- II says that $W^{fin} \subset W$ is dense for any rep W.
- $\mathbb{C}^{alg}[G] = C(G)^{fin} = L^2(G)^{fin}$ $(= C^\infty(G)^{fin} = \ldots\ldots)$ - common algebraic part, just as in case of $U(1)$.

Why? Suppose $f \in W \subset C(G)$ fin. dim subspace

$\Rightarrow$ let $f = f_1, \ldots f_n$ orthonormal basis of W

$$g \cdot f_i = \sum M_{ij}(g) f_j$$

$$\Rightarrow \quad f(g) = (g^{-1} f_1)(1) = \sum M_{j1}(g^{-1}) f_j(1)$$

$$= \sum \overline{M_{1j}(g)} \; f_j(1)$$

ie. f is a linear combo of the $\overline{M}_{1j}$ = matrix elements of g on $\overline{W} = W^*$.

So $C(G)^{fin} \subset \mathbb{C}^{alg}[G]$, converse obviously

- So I follows from II applied to $C(G)$ or $L^2(G)$.
- I $\Rightarrow$ II : Look at group algebra action, $C(G) \circlearrowright W$ action on any irrep by convolution - smeared combinations of group elements.

If $f \in \mathbb{C}^{alg}[G]$, $w \in W$

$$\Rightarrow \quad f * w \in W^{fin}$$

Why? $g \in G \Rightarrow g\cdot(f*w) = (gf)*w$
ie convolution action "extends" action of g.

But for $f \in C(G)^{fin}$ the functions gf all belong to some finite dim representation so all the $g\cdot(f*w)$ span a finite dim rep $\Longrightarrow f*w \in W^{fin}$.

Would like to take $f = \delta_1$, delta-function at the identity, so $f*w = w \in W^{fin} \Rightarrow W = W^{fin}$

– works if G finite, but $\delta_1 \notin C(G)$, not a good smeared combo of elements of G!

However can approximate δ_1 arbitrarily closely by continuous functions with $\int f = 1$, supported in shrinking nbhds of $1 \in G$.

$\Rightarrow$ identity operator is in the closure of the operators $f*\_$ for $f \in C(G)$

$\Rightarrow$ Use $C(G)^{fin} \subset C(G)$ dense

$\Longrightarrow$ can make $f*w$ arbitrarily close to w $\Rightarrow W^{fin} \subset W$ dense.

■ ($I \Rightarrow II$)

[for rest of Peter-Weyl refer to Segal].

Plancherel theorem (corollary):

$f \in \mathbb{C}^{alg}[G]$ $\quad f = \sum f_V \quad$ f_V = component in $V \otimes V^*$

&
$$\boxed{\|f\|^2 = \sum \frac{1}{\dim V}\|f_V\|^2} \quad (*)$$

$f \in L^2(G)$: $f = \sum f_V$, f_V square summable (ie RHS of $*$ converges) & $*$ holds

$f \in C^\infty(G)$ $\quad f = \sum f_V$ "rapidly decreasing"
$f \in C^\omega(G)$ real analytic $\quad f = \sum f_V$ "exponentially decreasing"

Same for any rep W : $W^{fin} \subset W$ dense, $W^{fin} = \bigoplus_{V \text{ irrep}} [W]_V$. If W a Hilbert space $\Rightarrow \|w\|^2 = \sum_V \|w_V\|^2$, but can now take lots of versions of W, W^{∞}, W^{ω}, etc where we require the w_V to be rapidly/exponentially decreasing etc – different completions of W^{fin}.

[Note Plancherel : $L^2(G) \simeq \ell^2\left(\begin{smallmatrix}\text{discrete set of}\\ \text{irreps } V\end{smallmatrix}\right)$]

Weyl Character Formula :

We know reps are determined by their character, characters are class functions, & in fact characters give orthonormal basis for class functions.

What *are* these characters? Any unitary matrix $g \in U(n)$ is diagonalizable $\Rightarrow$

class function determined by its value on diagonal matrices. e.g $SU(2)$ χ determined by $\chi|_{U(1) = \left(\begin{smallmatrix} a & \\ & a^{-1}\end{smallmatrix}\right) \subset SU(2)}$

Moreover $\chi\begin{pmatrix} q & \\ & q^{-1}\end{pmatrix} = \chi\begin{pmatrix} q^{-1} & \\ & q\end{pmatrix}$:

Weyl group symmetry, two diagonal matrices are conjugate in $U(n) \longleftrightarrow$ differ by a permutation.

So characters are an orthonormal basis for $L^2(\mathbb{T})^{\mathbb{Z}/2}$ ($L^2(T)^W$ for $T = \begin{pmatrix} \ddots \end{pmatrix}$, W = permutations)

~~P~~ Pw 9

What *are* these characters, really?

recall character is $\mathrm{tr}\,\pi(g) = \chi_\pi(g)$.

We're restricting to $\pi \subset SU(2)$, so

our rep $V = \bigoplus_{n\in\mathbb{Z}} [V]_n$ n-isotypic component

$\begin{pmatrix} q & \\ & q^{-1}\end{pmatrix} \cdot v_n = q^n v_n$.

So $\boxed{\chi_\pi\begin{pmatrix} q & \\ & q^{-1}\end{pmatrix} = \sum_{n\in\mathbb{Z}} \dim[V]_n \cdot q^n}$

For $SU(n)$ similarly $\chi_\pi\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix} = \sum_{\substack{i_1,\dots,i_n\in\mathbb{Z} \\ \sum i_j = 0}} \dim[V]_{i_1,\dots,i_n} \cdot a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n}$

$\prod a_i = 1$

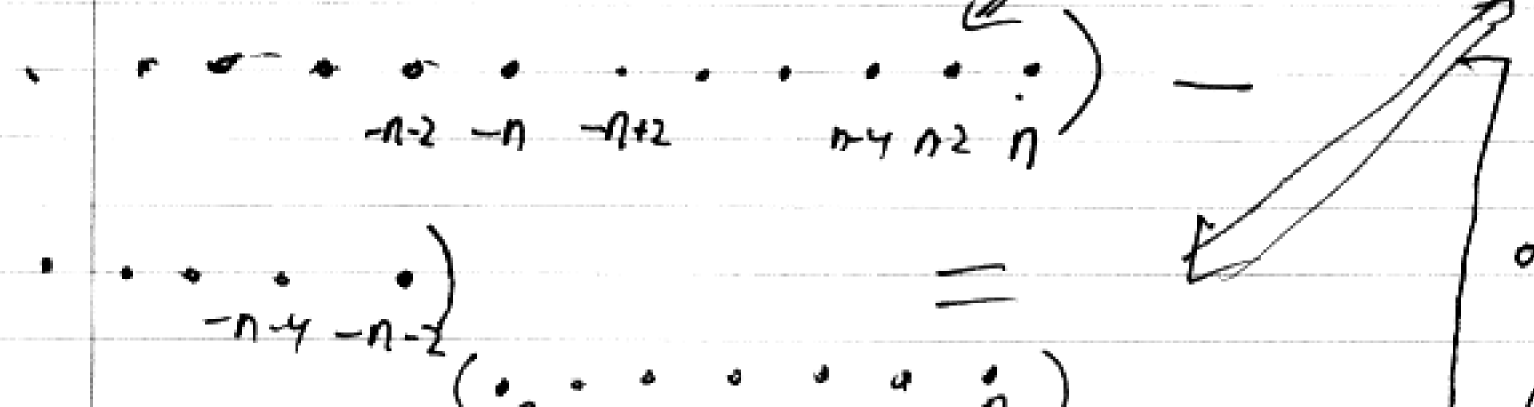
So character simply keeps track of all the eigenspaces of diagonal matrices & their dimensions.
- W symmetry of weights $\Longleftrightarrow$ W symmetry of character!

e.g $V^{(n)}$ $\underset{-n}{\cdot}\ \underset{-n+2}{\cdot}\ \cdot\ \cdot\ \cdots\ \underset{n}{\cdot}$

$$\boxed{\begin{aligned} \chi_{V^{(n)}}\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix} &= a^{-n} + a^{-n+2} + \dots + a^n \\ &= \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}} \end{aligned}}$$ WCF for SU(2)

$\left(\leftrightarrow \chi(\theta) = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} \qquad a = e^{i\theta} \right)$

Write suggestively: $\frac{q^{n+1}}{q-q^{-1}} - \frac{q^{-n-1}}{q-q^{-1}}$



i.e $\chi(\theta)$ is the difference of the characters of the Verma module V_n & V_{-n-2}:

$0 \to V_{-n-2} \to V_n \to V^{(n)} \to 0$

Weyl's integration $\qquad T = (\mathbb{C}^*)^n$ coords $q_1^{\pm 1}, \dots q_n^{\pm 1}$

Proposition f L^2 class function on $U(n) \Rightarrow$

$$\int_{U(n)} f(g)\, dx = \frac{1}{n!} \int_T f(q)\, \Delta\bar{\Delta}\, dq_1 \dots dq_n$$

where $\Delta = \prod_{i<j} (q_i - q_j)$, $\Delta\bar{\Delta} = \prod_{i<j} |q_i - q_j|^2$

$\Longleftrightarrow$ $\quad f, g$ L^2 class functions on $U(n) \Rightarrow$

$$\langle f, g \rangle_{L^2(G)} = \frac{1}{n!} \int_T f\, \bar{g}\, \Delta\bar{\Delta}\, dq_1 \dots dq_n = \frac{1}{n!} \langle \tilde{f}, \tilde{g} \rangle_{L^2(T)}$$

where $\tilde{f} = f \cdot \Delta \qquad \tilde{g} = g \cdot \Delta$

$$\Delta = \text{Vandermonde determinant} = \begin{vmatrix} 1 & \cdots & 1 \\ q_1 & \cdots & q_n \\ \vdots & & \\ q_1^{n-1} & \cdots & q_n^{n-1} \end{vmatrix}$$

Note: f is symmetric function of $q_1 \dots q_n$

$\tilde{f}$ is skew-symmetric " " "

Alternatively: $\langle f, g \rangle_{L^2(G)} = \langle f\Delta, g\Delta \rangle_{\{q_1 \le \dots \le q_n\} \subset T}$

- remove overcounting
by looking at Weyl chamber

Proposition says $\operatorname{vol}\left[G\begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix}\right] = |\Delta(q_1 \dots q_n)|^2$

Moral here $\displaystyle\int_{U(n)} F = \int_T \int_{\text{conj class}} H \, dq_1 \dots dq_n = \int_T |\Delta^2|\, H\, dq$

up to n!

More precisely: $T \times G/T \xrightarrow{\mu} G \qquad t, gT \longmapsto gtg^{-1}$

is generically $n!$ to 1 cover (on regular semisimple locus)

$$\int_G H = \frac{1}{n!} \int_{T \times G/T} J(q)\, \mu^* H_0 = \frac{\operatorname{vol}(G/T)}{n!} \int_T \mu^* H \cdot J(q)\, dq$$

Calculate $J(\xi)$: linearize $\mathfrak{t} \oplus \mathfrak{g}/\mathfrak{t} \xrightarrow{\sim} \mathfrak{g}$
$= \mathfrak{t} \oplus \mathfrak{t}^\perp$

At $q \in T$: $\xi \in \mathfrak{t}$ $\eta \in \mathfrak{t}^\perp$ at $(t,1) \in T \times G/T$

$$\frac{1}{t}\left[(1+\eta)\cdot\left(t(1+\xi)\right)(1-\eta) - t\right]$$
$$=\frac{1}{t}\left[t\xi + \eta t - t\eta\right]$$
$$= \xi + (t^{-1}\eta t - \eta) \in \mathfrak{t} \oplus \mathfrak{t}^\perp = \mathfrak{g}$$

$J(t) = \det\left(A(t^{-1}) - 1\right)$ $A(t^{-1})$ = action of t^{-1} on $\mathfrak{t}^\perp$

Diagonalize $A(t^{-1})$ in $\mathfrak{t}^\perp \otimes \mathbb{C}$: $t = (q_1 \dots q_n)$
basis E_{jk} $j \neq k$, e'value $q_k q_j^{-1}$

So $J(t) = \prod_{j \neq k} (q_k q_j^{-1} - 1) = \prod_{j<k} |q_j - q_k|^2$ ▨

e.g for S^3: area of sphere of all rotations with angle θ is $e^{i\theta} - e^{-i\theta}$.

Application to characters! χ character $\Rightarrow \tilde{\chi} = \Delta\chi$ skew symmetric on T
[characters w/o reps!]

$\chi \Leftrightarrow \tilde{\chi}$ finite Fourier polynomials.. try

$$\tilde{\chi}_{i_1,\dots,i_n} = \sum_{w \in W} (-1)^{\ell(w)} q_1^{w(i_1)} \cdots q_n^{w(i_n)}$$

for some $i_1, \dots, i_n$. give orthonormal basis for skew symmetric fns

$$\Rightarrow \chi = \frac{\tilde{\chi}}{\Delta} = \frac{\sum_w (-1)^w q_1^{w(i_1)} \cdots q_n^{w(i_n)}}{\prod (q_i - q_j)}$$

$SU(2)$: $\dfrac{q^n - q^{-n}}{q - q^{-1}} = \chi_{V(n-1)}$.

Application 2D YM / random matrices: $L^2(G)^G$
realized as n ~~free~~ fermions on the circle!
eivals behave as free fermions

Def V rep of Lie group G, $v \in V$ is **smooth** if $g \longmapsto \pi(g)\cdot v$ is a smooth map $G \longrightarrow V$

- **analytic** if $G \longrightarrow V$ is analytic

$V \supset V^\infty \subset C^\infty(G,V)$ space of smooth vectors

Claim $V^\infty \subset C^\infty(G,V)$ is a closed subspace, $G \times V^\infty \to V^\infty$ continuous

Pf : to calculate $\lim v_i$ $v_i \in V^\infty$ write $f_i(g) = \pi(g) v_i$ corresponding functions, ~~take~~ we are assuming $\lim f_i = f \in C^\infty(G,V)$ exists.
Take $v = f(e) \in V^\infty$, & $f(g) = \pi(g) v \Rightarrow v \in V^\infty$. ∎

Now V^∞ is acted on by $\mathfrak{g}$ ($\Rightarrow U\mathfrak{g}$)
via $X \in \mathfrak{g} \mapsto \pi(x) v = \frac{d}{dt}\Big|_{t=0} (e^{tX} v)$

$$= \lim_{t \to 0} \frac{e^{tx} v - v}{t} \in V^\infty$$

$()^\infty$: G-reps $\longrightarrow$ smooth G-reps

Theorem $V^\infty \subset V$ is dense.

Proof

- **Lemma** $W \subset V$ subspace s.t. $\forall x \in \mathfrak{g}, w \in W$

$$\pi(X) w := \lim_{t \to 0} \frac{\pi(\exp tx) v - v}{t} \in W \Longrightarrow$$

$$W \subset V^\infty$$

Pf Consider $f_w : G \longrightarrow V$ $f_w(g) = \pi(g)\cdot w$

We know by ~~as~~ hypothesis f_w is differentiable at $e \in G$.
Hence at any g: $\lim_{t \to 0} \frac{f_w(\exp(tx)g) - f_w(g)}{t}$

$x^g = g^{-1} x g$

$$= \pi(g) \lim_{t \to 0} \frac{\pi(g^{-1} \exp tx\, g) w - w}{t} = \pi(g) \left(\pi(x^g) \cdot w\right)$$

X^g depends smoothly on $g \Rightarrow f_w \in C^1$.

Next: $X\cdot(X\cdot w) \in W$ as well $\Rightarrow$ second derivatives $\exists$
& keep going $\Rightarrow$ $f_w \in C^\infty$. ■

Lemma 2 $C_c^\infty(G) \times V \longrightarrow V^\infty$

(analog of $C^{alg}(K) \times V \longrightarrow V_K$ K-finite)

Proof $f \in C_c^\infty(G)$, $v \in V$, $X \in \mathfrak{g}$

$$w = f * v = \int_G f(g)\,\pi(g)\cdot v \, dg$$

$$\pi(X)\cdot w = \lim \frac{\pi(\exp tX)w - w}{t}$$

$$= \lim t^{-1} \int_G f(g)\big(\pi(\exp tX)\pi(g) - \pi(g)\big)v \, dg$$

$$= \lim t^{-1}\left(\int_G f(\exp(-tX)g)\,\pi(g)v \, dg - \int f(g)\pi(g)v\, dg\right)$$

$$= \lim t^{-1}\left(\int_G \big(f(\exp(-tX)g) - f(g)\big)\pi(g)\, v \, dg\right.$$

$$= \lim -\int_G \frac{f(\exp(-tX)g) - f(g)}{-t}\,\pi(g)\, v \, dg$$

$$= -\int_G (X\cdot f)(g)\,\pi(g)\, v\, dg \in \mathrm{Im}\big(C_c^\infty G \times V \to V\big)$$

So by Lemma 1 $\mathrm{Im}(C_c^\infty G \times V \to V) \subset V^\infty$. ■

To prove theorem show $(C_c^\infty G)V \subset V$ dense!

density of C^∞ fns. explicitly $v \in V$ $\varepsilon > 0$, $|\ |$ any seminorm

$S = \{g \in G : |\pi(g)v - v| < \varepsilon\}$, $f \in C_c^\infty G$ s.t. $f \geq 0$,

supp $f \subset S$, $\int_G f(g) = 1$ $\Rightarrow$

$$|\pi(f)v - v| \leq \int_G f(g)\,|\pi(g)v - v| \, dg < \varepsilon$$ ■

Towards Harish-Chandra Modules

Principal series for $SL_2\mathbb{R}$

$$SL_2\mathbb{R} \supset B_\mathbb{R} = \left\{ \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \right\} \twoheadrightarrow T_\mathbb{R} = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right\} \longrightarrow \mathbb{C}^*$$

$$T_\mathbb{R} \simeq \mathbb{R} \times \mathbb{Z}/2 \qquad a \longmapsto |a|^s \qquad s \in \mathbb{R}$$

$$M_s = \mathrm{Ind}_{B_\mathbb{R}}^{SL_2\mathbb{R}} \mathbb{C}_s = \{ f \in \mathrm{Fun}\, SL_2\mathbb{R} : f(gb) = \chi_s(b)\, f(g) \}$$

$$SL_2\mathbb{R}/B_\mathbb{R} = \mathbb{RP}^1 \longleftarrow \mathbb{R}^2 \setminus 0 = SL_2\mathbb{R}/N_\mathbb{R}$$

Functions on $\mathbb{R}^2 \setminus 0$ "homog of degree s"

In particular $f(-x) = f(x)$ $x \in \mathbb{R}^2 \setminus 0$

Can identify $M_s \simeq \mathrm{Fun}($~~$\mathbb{RP}^1 \supseteq$~~$S^1 \subset \mathbb{R}^2 \setminus 0)$

... extend uniquely to all $\mathbb{R}^2 \setminus 0$.

... topologically line bundle

$$\begin{tikzcd} SL_2\mathbb{R} \times_{B_\mathbb{R}} \mathbb{C}_s \arrow[d] \\ \mathbb{RP}^1 \end{tikzcd}$$

is trivial

(could get Möbius line bundle on $\mathbb{RP}^1$ by looking at characters of $T_\mathbb{R} = \mathbb{R} \times \mathbb{Z}/2$ nontrivial on $\mathbb{Z}/2$)

What do we mean by "$\mathrm{Fun}\, S^1$" ?

$S^1 = \mathbb{T}$, can study $\mathrm{Fun}\, S^1$ via Fourier series

...ie rep theory of $\mathbb{T}$,

$K = SO_2 = \mathbb{T} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \subset SL_2\mathbb{R}$ maximal compact subgroup

So can look at $(M_s)_K$ K finite vectors!

finite Fourier series

Problem! $(M_s)_K$ not preserved by $G_\mathbb{R}$!

$\Rightarrow$ pass to Harish-Chandra modules

Principal Series' : $SL_2\mathbb{R} \circlearrowright L^2(\mathbb{R})$ via

$g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad (g\cdot f)(x) = |cx+d|^s f(g^{-1}x)$

or $\quad (g\cdot f)(x) = \mathrm{sgn}(cx+d)\cdot|cx+d|^s f(g^{-1}x)$

$SL_2\mathbb{C}$ principal series : $\quad B = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$

$a \in \mathbb{C}^* = \mathbb{R}_+ \times S^1$: characters given by $s \in \mathbb{R}, n \in \mathbb{Z}$

$\Longleftrightarrow \quad \lambda, \mu \in \mathbb{C}$ with $\lambda - \mu \in \mathbb{Z}$

$z \longmapsto z^\lambda \bar{z}^\mu$

[... holomorphic characters $\longleftrightarrow$ characters of S^1

" reps of $SL_2\mathbb{C} \longleftrightarrow$ reps of SU_2]

$\Rightarrow$ can induce: $\mathrm{Ind}_B^G \mathbb{C}_{\lambda,\mu} = \{ f \in L^2(SL_2\mathbb{C}) : f(gb) = \chi_{\lambda,\mu}(b)\cdot f(g) \}$

L^2 sections of line bundles.

- $(\mathfrak{g},K)$-modules $\mathfrak{g}$ Lie algebra, $\mathfrak{k} = \mathrm{Lie}\,K \subset \mathfrak{g}$
  & K acts on $\mathfrak{g}$ extending adjoint action on $\mathfrak{k}$

  ⇒ M $\mathfrak{g}$-module & K-module s.t. $\mathfrak{k}$ acting from both agree & $k(\xi v) = (\mathrm{Ad}\,k \cdot \xi)\, kv$ $\forall k \in K, \xi \in \mathfrak{g}$

- admissible : $M = \bigoplus_{\lambda \in K^\wedge} [M]_\lambda$ [illegible] ⊗ · finite dimensional

  ... each K-type appears finitely many times.

$$\begin{tikzcd} & G & \\ G_{\mathbb{R}} & & K_{\mathbb{C}} \\ & K & \end{tikzcd}$$

We'll apply to $\mathfrak{g}_{\mathbb{C}}$ $\mathbb{C}$-Lie algebra, K = max compact of $G_{\mathbb{R}}$. ⇒ M is module for $\mathfrak{g}_{\mathbb{C}}$ & $K_{\mathbb{C}}$ (holomorphic for latter).

Idea: K contains all topology of $G_{\mathbb{R}}$ (deformation retracts to K), & its rep theory is "combinatorial" ... contains info about disconnected groups etc.
Passing to $\mathfrak{g},K$ modules forgets topology, gets rid of phenomena like π_i irred $\pi_1 \hookrightarrow \pi_2$ dense but not isomorphism – doesn't occur for Hilbert spaces :

Theorem • V irred unitary rep of G ⇒ V_K^∞ is admissible
• Unitaries full subcategory of admissible $\mathfrak{g},K$-modules : isom of unitaries ⟺ infinitesimal equivalence

Theorem V admissible ⇒ V_{K_R} consists of analytic vectors : $G \longrightarrow V$, $g \longmapsto g.v$ analytic. $\boxed{\text{So } \underline{\underline{V_K^\infty}} = V_K}$

Corollary V admissible ⇒ HC gives bijection
{closed $G_{\mathbb{R}}$-subreps of V} ⟷ {$(\mathfrak{g},K)$-submodules of V_K}

Def H-C module : admissible f.gen $(\mathfrak{g},K)$-module

Corollaries 1. V fin dim continuous rep of G $\Rightarrow$ V is smooth.

2. $G_{\mathbb{R}} \supset K_{\mathbb{R}}$ Lie group, max compact (π, V) rep of $G_{\mathbb{R}}$ $\Rightarrow$ $(V^{\infty})_{K_{\mathbb{R}}}$ K-finite smooth vectors is dense in V, and acted on by $\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{R}}$

Proof: $v \in V^{\infty}_{K_{\mathbb{R}}}$, $V_v = K_{\mathbb{R}} \cdot v$ finite dimensional

$\pi(\mathfrak{g}_{\mathbb{R}}) \cdot V_v$ is stable under $\pi(k_{\mathbb{R}})$:

$$\underset{\in k_{\mathbb{R}}}{\pi(k)}\ \underset{\in \mathfrak{g}_{\mathbb{R}}}{\pi(\xi)}\ \underset{\in V_v}{v'} = \underbrace{\pi(\xi)\big(\underset{\in V_v}{\pi(k)\cdot v}\big) + \pi([k,\xi]) \cdot \underset{\in V_v}{v'}}_{\in\ \pi(\mathfrak{g}_{\mathbb{R}})\cdot V_v}.$$

$\Rightarrow$ $\pi(\mathfrak{g}_{\mathbb{R}}) V_v$ is $K_{\mathbb{R}}$-stable

$\Rightarrow$ $\pi(\xi) v$ is $K_{\mathbb{R}}$-finite ■

In fact $V^{\infty}_{K_{\mathbb{R}}}$ is a $\mathfrak{g}_{\mathbb{C}}$-module & an algebraic $K_{\mathbb{C}} = K_{\mathbb{R}} \otimes \mathbb{C}$ module:

Algebraic $K_{\mathbb{C}}$-modules $\longleftrightarrow$ $\bigoplus$ fin dim $K_{\mathbb{R}}$-modules.

Def $(\mathfrak{g}, K)$-module $\mathfrak{g}$ Lie algebra

Proof of Corollary V admissible $\Rightarrow HC(V) = V_{K_{\mathbb{R}}}$

$W \subset V$ closed $\Rightarrow \overline{W_{K_{\mathbb{R}}}} = W$

So need: $M \subset V_{K_{\mathbb{R}}}$ H-C submodule $\Rightarrow \overline{M} \subset V$ is $G_{\mathbb{R}}$-invariant. To check $\pi(g) \cdot M \subset \overline{M} \iff$
$\langle \lambda, \pi(g) M \rangle = 0 \quad \forall \lambda \in M^{\perp} \subset V^*$

But $g \mapsto \langle \lambda, \pi(g) m \rangle$ analytic function on $G_{\mathbb{R}} +\epsilon$
$\Rightarrow$ check it vanishes on all derivatives at $e \in G$
$\iff$ look at $U\mathfrak{g}$ action. But M is $U\mathfrak{g}$ invariant. ■

So eg irred. admissible $G_{\mathbb{R}}$-mod $\iff$ irred H-C module.

$SL_2\mathbb{R}$ $\mathfrak{g}, K$ modules:

$|\alpha|^2 - |\beta|^2 = 1$

~~[illegible]~~

Use $\underset{\text{"Aut } \mathbb{H}\text{"}}{SL_2\mathbb{R}} \simeq \underset{\text{"Aut } D\text{"}}{SU(1,1)} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$

via conjugation by $z \mapsto \frac{z - i}{z + i}$

$h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \quad e = \begin{pmatrix} & 1 \\ & \end{pmatrix} \quad f = \begin{pmatrix} & \\ 1 & \end{pmatrix}$

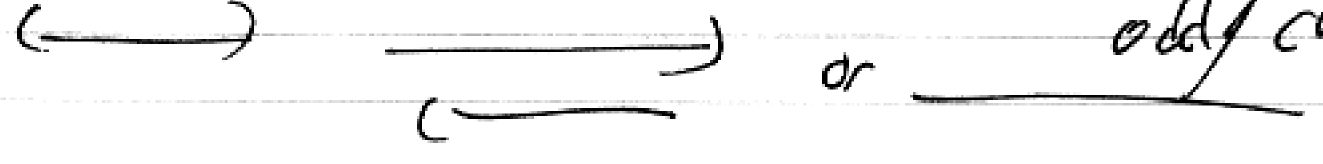
$K = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\}$ ⊕ $\quad ih$ is infi generator

So set $M = \bigoplus_{\mathbb{Z}} M_n$ K-type decomposition

& e, f act as ladder operators $M_n \underset{f}{\overset{e}{\rightleftarrows}} M_{n+2}$

M irreducible $\Rightarrow M_n$ 0 or 1 dim even/odd or

$\Rightarrow$



Harish-Chandra Modules for $SL_2\mathbb{R}$

$(\mathfrak{sl}_2\mathbb{R}, SO(2))$ modules : $SO(2) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

More convenient basis : want e, f, h compatible with $SO(2)$.

Use $\underset{\text{Aut } \mathbb{H}}{SL_2\mathbb{R}} \simeq \underset{\text{Aut } D}{SU(1,1)} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$, $|\alpha|^2 - |\beta|^2 = 1$

· Conjugate by $z \longmapsto \frac{z-i}{z+i}$ $\quad i \mapsto 0 \quad \infty \mapsto 1 \quad 0 \mapsto -1$

Infinitesimally take $H = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad F = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$

$E = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$

$[H, X] = 2X \quad [H, Y] = -2Y, \quad [X, Y] = H$

$\exp tH = SO(2)$

So on $(\mathfrak{sl}_2\mathbb{C}, SO_2)$ module M will have H acting with integer eigenvalues (K-type)

$$M = \bigoplus_{\mathbb{Z}} M_n \qquad M_n \underset{F}{\overset{E}{\rightleftarrows}} M_{n+2}$$

Where to start? no obvious top or bottom!

Use Casimir $C = 2XY + 2YX + H^2 \in U\mathfrak{g}$
generator of center, acts by scalar on irreducibles

$\mathrm{Mod}_{(\mathfrak{sl}_2\mathbb{C}, SO_2)} \longrightarrow \mathrm{Mod}\, \mathbb{C}[C]$ = sheaves on the line

no ~~intera~~ interactions between fibers $\mathrm{Mod}^{C=\lambda}_{\mathfrak{g},K}$ for different λ ... only thing we could have is C acting by Jordan blocks, let's ignore. $\Longrightarrow$

Describe $\mathrm{Mod}^{C=\lambda^2-1}_{\mathfrak{g},K}$ $\quad$ normalization

[Recall : on H-W rep $V^{(n)}$ $(2XY+2YX-H^2)v_n$

$= \beta n^2 v_n + 2[X,Y]v_n = (n^2-2n)v_n$ have quadratic choice]

Rewrite $C = 4XY - 2H + H^2$

$$\underset{\lambda^2 v}{\overset{\shortparallel}{(C+1)v}} = 4XYv + (n^2-2n+1)v$$

$$\Rightarrow XYv = \frac{1}{4}\left((n-1)^2 - \lambda^2\right)v$$

$$YXv = \frac{1}{4}\left((n+1)^2 - \lambda^2\right)v$$

So XY invertible unless $\lambda = \pm(n-1)$
YX " " $\lambda = \pm(n+1)$

- $\lambda \notin \mathbb{Z}$ get XY, YX always invertible, so
  $\mathrm{Mod}_{g,K}^{\lambda^2-1} \simeq \mathrm{Vect} \oplus \mathrm{Vect}$:

  take M_{2n}, M_{2n+1} each freely generates a chain (even, odd) of vector spaces, rep completely determined.
  So two irreducibles, M_λ^{even}, $M_\lambda^{\lambda,\text{odd}}$ everything else direct sum.

- $\lambda \in \mathbb{Z} \setminus 0$: again can decompose into even/odd blocks $\mathrm{Mod}^{\text{even}}$, $\mathrm{Mod}^{\text{odd}}$

  have no interaction, every $M = M^{\text{even}} \oplus M^{\text{odd}}$

Assume λ odd $\Rightarrow$ ~~[illegible]~~ ~~[illegible]~~ even λ

have $\mathrm{Mod}^{\text{odd},\lambda} = \mathrm{Vect}$:
$$\begin{tikzcd} \bullet & \bullet \arrow[r, bend right, "X"'] & \bullet \arrow[l, bend right, "Y"'] \arrow[r, bend right] & \bullet \arrow[l, bend right] & \bullet \end{tikzcd}$$

XY, YX invertible on all odd V_n

Even reps are more interesting:

Three vector spaces to consider

$$\begin{tikzcd} C \arrow[r, bend right, "s"'] & A \arrow[l, bend right, "r"'] \arrow[r, bend right, "p"'] & B \arrow[l, bend right, "q"'] \end{tikzcd}$$

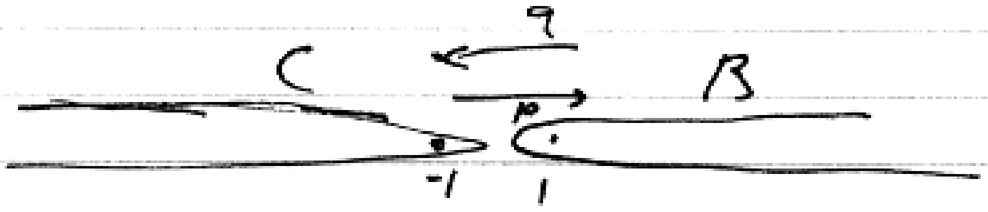
$rs = sr = 0$
$qp = pq = 0$

Category of reps of a quiver $\bullet \rightleftarrows \bullet \rightleftarrows \bullet$ with relations.

Irreducible reps: only one vector space nonzero

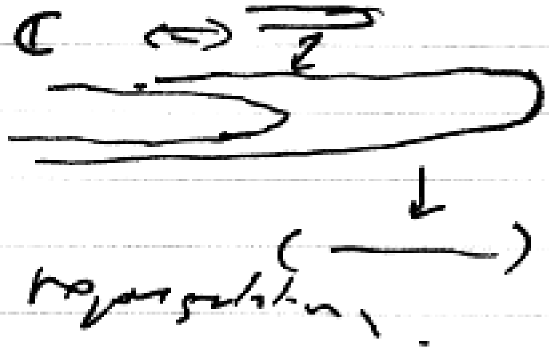
$\Downarrow$ Verma $V_{-\lambda-1}$ $\Downarrow$ $V^{(\lambda)}$ fin dim dual Verma $V^*_{\lambda+1}$

$\lambda = 0$



2 irreducibles $qp = pq = 0$

So representations form interesting categories! have reps which are indecomposable but not irreducible: e.g. $\mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C} \iff$



Verma module V_n $n \in \mathbb{Z}_+$
- nontrivial extensions between representations.

Can sometimes model with quivers: basically generators & relations... more generally: model by geometry, in our case

$SL_2\mathbb{R} \circlearrowright \mathbb{CP}^1$

$SO_2\mathbb{C} \circlearrowright \mathbb{P}^1$
$\cong$
$\mathbb{C}^*$

$(\mathfrak{sl}_2\mathbb{R}, SO_2\mathbb{R})$ $(\mathfrak{sl}_2\mathbb{C}, SO_2\mathbb{C})$

Note $g^{-1} = \cancel{\frac{ax-c}{-bx+d}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ $\quad g^{-1}x = g^{-1}\begin{pmatrix} 1 \\ x \end{pmatrix} = \frac{ax-c}{-bx+d}$

$(g^{-1})' = \frac{1}{(-bx+d)^2}$ since $ad-bc=1$. $\Rightarrow \boxed{-\frac{s}{2} \text{ densities}}$

Now use $K = SO_2 \twoheadrightarrow \mathbb{RP}^1$: $SL_2\mathbb{R} = KMAN$

$$V_{b,s} \simeq \{ \tilde{F} : K \to \mathbb{C} : \tilde{F}(km) = b(m)\tilde{F}(k) \}$$

$$(K \cap B = M \quad \ldots \quad K \simeq S^1 \subset \mathbb{R}^2 - 0 = SL_2\mathbb{R}/N)$$

$$\overset{''}{SL_2\mathbb{R}/AN}$$

So as K-rep $V_{b,s} = \begin{cases} \bigoplus_{n \text{ odd}} \mathbb{C}_n & b = \operatorname{sgn} \\ \bigoplus_{n \text{ even}} \mathbb{C}_n & b = 1 \end{cases}$

later

Need to calculate $\mathfrak{sl}_2$ action on basis :

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = \begin{bmatrix} 1 \\ x \end{bmatrix} \Rightarrow x = -\tan\theta, \quad \theta = -\arctan x$$

$$d\theta = \frac{-dx}{1+x^2} \qquad |d\theta| = \frac{|dx|}{1+x^2}$$

[Basis for $\mathbb{C}_n$ is $e^{in\theta}$

$\longleftrightarrow \quad f_n(x) = e^{in\arctan x}(\cancel{x}+x^2)^{s/2}\left(\cdot |dx|^{-s/2}\right)$]

$e = \begin{pmatrix} & 1 \\ & \end{pmatrix} \quad \exp(te)^{-1} = \begin{pmatrix} 1 & -t \\ & 1 \end{pmatrix} \quad \ldots \quad x \sim \begin{pmatrix} 1 \\ x \end{pmatrix}$

$$(\exp(te)\cdot f)(x) = f\left(\frac{x}{-tx+1}\right) \underbrace{b(\operatorname{sgn}(-tx+1))}_{\overset{''}{1} \text{ for } t \text{ small}} \underbrace{|-tx+1|^s}_{\overset{''}{(-tx+1)^s} \; t \text{ small}}$$

$$\Rightarrow (e\cdot f)(x) = f'(x)\cdot x^2 - sx f(x)$$

$$\boxed{e = x^2 \frac{d}{dx} - sx}$$

$(\exp(tf)\cdot f)(x) = f(x-t) \Rightarrow$

$$\boxed{f = -\frac{d}{dx}}$$

$\exp(th) = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}$

$$(\exp(th)\cdot f)(x) = f\left(\frac{e^t x}{e^{-t}}\right) e^{-ts}$$

$$\boxed{h = 2x\frac{d}{dx} - s}$$

$s=0$ get usual action $f = -\frac{d}{dx}$, $e = x^2\frac{d}{dx}$, $h = 2x\frac{d}{dx}$

- we're trivializing $-\frac{s}{2}$ densities by ~~$\left(\frac{dx}{1+x^2}\right)^{-s/2}$~~ , $|dx|^{-s/2}$ which is translation invt hence f doesn't change

$\mathfrak{sl}_2$ acts on $-\frac{s}{2}$ densities by Lie derivative

Calculate Casimir $C = 2ef + 2fe + h^2 =$

$$= 2(x^2\partial - sx)(-\partial) + 2(-\partial)(x^2\partial - sx) + (2x\partial - s)^2$$

$$= s^2 + 2s$$

Let $\lambda = s+1 \implies C = \lambda^2 + 1$ as before.

$\implies s \notin \mathbb{Z}$ $V_{\lambda,s}$ irreducible.

Otherwise must calculate quiver:

$H = i\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = i(e-f) \longmapsto i(x^2\frac{d}{dx} - sx + \frac{d}{dx})$

$Y = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$ .....

Check $Hf_n = i(-inf_n + sxf_n - sxf_n) = nf_n$

$(\frac{d}{dx} f_n = \frac{-in}{x^2+1} f_n + \frac{sx}{x^2+1} f_n)$

Lemma $\boxed{Xf_n = -\frac{n-s}{2} f_{n+2} \qquad Yf_n = \frac{n+s}{2} f_{n-2}}$

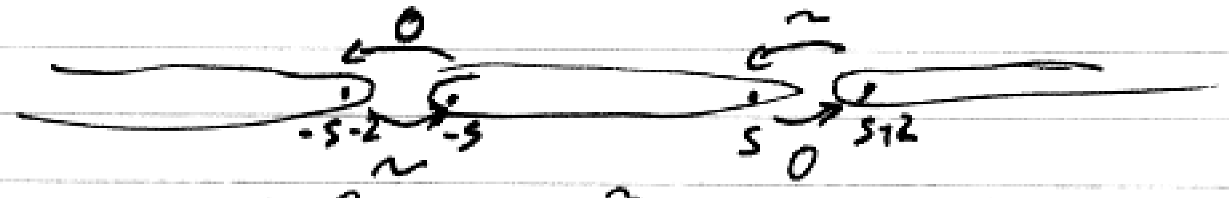
$\implies s = -1$:

$$\begin{tikzcd} \mathbb{C} \arrow[r, shift right, "0"'] & \mathbb{C} \arrow[l, shift right, "0"'] \end{tikzcd}$$

two irreducibles

ξ-fn! ↩ $(-\frac{s}{2} = \frac{1}{2})$

$s > -1$:



$$\begin{tikzcd} \mathbb{C} \arrow[r, shift right, "\sim"'] & \mathbb{C} \arrow[l, shift right, "0"'] \arrow[r, shift right, "0"'] & \mathbb{C} \arrow[l, shift right, "\sim"'] \end{tikzcd}$$

so  is a sub

$s < -1$:

$$\begin{tikzcd} \mathbb{C} \arrow[r, shift right, "0"'] & \mathbb{C} \arrow[l, shift right, "\sim"'] \arrow[r, shift right, "\sim"'] & \mathbb{C} \arrow[l, shift right, "0"'] \end{tikzcd}$$

 is a quotient.

Note duality $s \longleftrightarrow -2-s$ -1 ↺

Serre duality : $\omega = \mathcal{O}(-2)$ $E \longrightarrow E^*$

Principal Series & Harish-Chandra Modules

$SL_2\mathbb{R} \supset B_{\mathbb{R}} = MAN$ $\quad M = \pm Id \quad A = \begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\ a>0$

$N = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$

δ character of M = sgn or triv

s character of A $\quad a \mapsto a^s \quad s \in \mathbb{C}$

$$V_{\delta,s} = Ind_{B_{\mathbb{R}}}^{SL_2\mathbb{R}} \mathbb{C}_{\delta,s} = \{F : SL_2\mathbb{R} \to \mathbb{C} : F(gman) = F(g)\delta(m)a^s\}$$

= sections of bundle on $\mathbb{RP}^1 = SL_2\mathbb{R}/B_{\mathbb{R}}$

$T_\ell \mathbb{P}^1 = Hom(\ell, \mathbb{C}^2/\ell)$
$= Hom(\ell, \ell^*)$
$= (\ell^*)^2$

associated to $\quad SL_2\mathbb{R}/N_{\mathbb{R}} = \mathbb{R}^2 - 0 \quad$ principal bundle

$\downarrow MA \qquad$ for $MA = \mathbb{R}^*$

$\mathbb{RP}^1 \qquad$ - ie real line bundle

Tautological line bundle. In complex case this is $\mathcal{O}(-1)$. $\mathcal{O}(2)$ is tangent bundle, so sections of $\mathcal{O}(-1)$ are $\frac{1}{2}$ forms

linear fns on $\mathcal{O}(-1)$ are $-\frac{1}{2}$-forms, so associated bundle for sgn, s is $-\frac{s}{2}$-forms, triv, s is $-\frac{s}{2}$-densities:

s-form: expression $f(x)\,dx^s$, transforms under chg of coords as $g\cdot(f(x)\,dx^s) = f(g^{-1}x)(d(g^{-1}x))^s$
$= f(g^{-1}x)((g^{-1})')^s\,dx^s$

s-density: expression $f(x)\,|dx|^s$,
$g\cdot(f(x)\,dx^s) = f(g^{-1}x)\,|(g^{-1})'|^s\,dx^s$.

Concretely: Let $\bar{N} = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}$. $\bar{N}MAN \subset SL_2\mathbb{R}$ dense open: inverse image of $\mathbb{R} \subset \mathbb{RP}^1$ open $\bar{N}$ orbit (cf $\mathbb{C} \subset \mathbb{CP}^1$).

$V_{\delta,s} \ni F \longmapsto f(x) = F\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ fn on $\mathbb{R}$. $\quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$(g\cdot f)(x) = F\left(g^{-1}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) = F\left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}\right)$$

$$= F\begin{pmatrix} d-bx & -b \\ -c+ax & a \end{pmatrix} = F\left(\begin{pmatrix} 1 & 0 \\ \frac{ax-c}{-bx+d} & 1 \end{pmatrix}\begin{pmatrix} d-bx & -b \\ 0 & \frac{1}{d-bx} \end{pmatrix}\right)$$

$$= F\left(\begin{pmatrix} 1 & 0 \\ \frac{ax-c}{-bx+d} & 1 \end{pmatrix}\right)\cdot \delta(sgn(d-bx))\,|d-bx|^s = f(g^{-1}x)\,\delta(sgn(d-bx)) \cdot |(g^{-1})'|^{-s/2}$$

Corollary: (Subrepresentation theorem) Every irreducible H-C module appears as a sub of a principal series representation.

How to realize pieces?

$\mathbb{Z} \ni s > -1$ have holomorphic induction

$\mathrm{Ind}_B^{SL_2\mathbb{C}} \mathbb{C}_s \simeq$ homogeneous polynomials of degree s

… can restrict to $\mathbb{R}P^1 \subset \mathbb{C}P^1$,

get homog polynomials of degree s on $\mathbb{R}P^1 \subset V_{b,s}$.

<u>Discrete series</u> $\mathrm{Stab}(i) \subset SL_2\mathbb{R}$ is $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$\frac{ai+b}{ci+d} = i$: $ai+b = di-c$ $a=d$ $b=-c$ ~~and~~ $a^2+b^2=1$.

$\Rightarrow \mathbb{H} = SL_2\mathbb{R}/K$

$\frac{a(-i)+b}{c(-i)+d} = -i \Rightarrow -ai+b = -di+c$. $\mathbb{H}^- = SL_2\mathbb{R}/K$ also.

Given character $n \in \mathbb{Z}$ of K $\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} \mapsto e^{in\theta} \in U(1)$

$\Rightarrow$ define holomorphic induction $\mathrm{Hol}(\mathrm{Ind}_K^{SL_2\mathbb{R}} \mathbb{C}_n \subset \left\{ \begin{array}{l} F: SL_2\mathbb{R} \to \mathbb{C} \\ F(gk) = \chi_n(k) F(g) \end{array} \right\}$

$=$ ~~{f holomorphic on $\mathbb{H}$ in~~ [illegible] ~~on $\mathbb{H}$~~

impose holomorphy on F:

F is a section of line bundle on $\mathbb{H}$:

$SL_2\mathbb{R}$ is double cover of $PSL_2\mathbb{R}$ = unit tangent bundle of $\mathbb{H}$, n-th character $\longleftrightarrow$ $+\frac{n}{2}$ forms

… look at $\{ f(z)\, dz^{+\frac{n}{2}} : f \text{ holomorphic} \}$

$g \in SL_2\mathbb{R}$ acts by $f(g^{-1}z)\, (dg^{-1}(z))^{+\frac{n}{2}}$

$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ $= f\left(\frac{dz-c}{-bz+d}\right) \cdot (-bz+d)^{-n}\, dz^{+\frac{n}{2}}$

$\Longleftrightarrow D_n = \{ f \in \mathrm{Hol}(\mathbb{H}) \}$ $g \cdot f(z) = f(g^{-1}z)(-bz+d)^{-n}$

$\bar{D}_n$ complex conjugate $f \in \mathrm{Antihol}(\mathbb{H})) \simeq \mathrm{Hol}(\bar{\mathbb{H}})$

Series for $SL_2(F)$

F a field - e.g. $\mathbb{R}$, $\mathbb{F}_q$, $\mathbb{Q}_p$ [$\mathbb{C}((t))$?]

$SL_2(F) \supset \overset{\text{(tori)}}{\text{Cartan subgroups}}$: $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ conjugate over $\bar{F}$ to $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$

eg split torus : $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \simeq F^\times$

In fact ~~with it~~ suffices to go to a quadratic extension $E = F(\sqrt{D})$ to split g.

$\Rightarrow$ assign torus to D, $\begin{pmatrix} x & y \\ Dy & x \end{pmatrix}$ with $x^2 - Dy^2 = 1$

"unit circle"

.... in $GL_2(F)$ just take $\begin{pmatrix} x & y \\ Dy & x \end{pmatrix}$: $x^2 - Dy^2 \neq 0$ always (D not squared

ie $E^\times \subset GL_F(E)$ itself.

To split torus assign principal series : $\chi : F^\times \longrightarrow \mathbb{C}^\times$

multiplicative character $\Rightarrow \mathrm{Ind}_B^G \, \mathbb{C}_\chi$ $\qquad B \to F^\times \longrightarrow \mathbb{C}^\times$

To nonsplit tori : assign reps for characters of T_D

$D \in F^\times/(F^\times)^2$ discrete series

eg. $\mathbb{F}_p$ have one discrete series attached to $\mathbb{F}_{p^2}^\times$

$\mathbb{Q}_p$ have three discrete series : $D = p$, $D = \sqrt[p-1]{1}$,

$D = p\sqrt[p-1]{1}$

Circles all compact ... discrete series

$\mathbb{Q}_p^\times \simeq \mathbb{Z} \times \mathbb{Z}_{p-1} \times \{1 + a_1 p + a_2 p^2 + \dots\}$

$\updownarrow$ $\quad \leftarrow$ Teichmüller lifts of $\mathbb{F}_p$ $\rightarrow$

$\mathbb{R}^\times \simeq \mathbb{Z}/2 \times \mathbb{R}_+$ $\qquad \mathbb{F}_p^\times \simeq \mathbb{Z}_{p-1}$

Drinfeld upperhalf plane / Deligne-Lusztig variety:

Given two lines $\ell_1, \ell_2 \in \mathbb{P}^1$ can ask if $\ell_1 = \ell_2$ or $\ell_1 \neq \ell_2$ (only two G-invariant questions)

$\ell_1 \neq \ell_2 \iff \ell_2 \in B_{\ell_1} \cdot (w \ell_1)$ open orbit of Stab ℓ_1

Given two flags $B_1, B_2 \in G/B \Rightarrow W$ questions to ask: $B_1 \underset{w}{—} B_2$ if $[B_2] \in B_1 w B_1$ mod B_1

Real version: $\mathcal{B}_{\mathbb{R},w} = \{ B \in \mathcal{B} : B \underset{w}{—} \overline{B} \}$

Flags in given position with $\overline{B}$ w order two

$G_{\mathbb{R}} = (G_{\mathbb{C}})^{-,\text{invol}} \circlearrowright \mathcal{B}_{\mathbb{R},w}$

e.g. $SL_2\mathbb{R} \circlearrowright \mathcal{B}_{\mathbb{R},w} = \mathbb{P}^1 \setminus \mathbb{R}\mathbb{P}^1$: ~~flags~~ lines that aren't real

Each $\mathcal{B}_{\mathbb{R},w}$ assigned to a particular kind of torus: $T_{\mathbb{C}}^{w\gamma}$ fixed points of $z \mapsto (\bar{z})^w$

$SL_2\mathbb{C}$: $U(1) = (\mathbb{C}^\times)^{-w}$ $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$\mathbb{R}^\times = (\mathbb{C}^\times)^{-}$

A version $T_{\mathbb{R},w}$ will be $\subset$ Stabilizer of point in $\mathcal{B}_{\mathbb{R},w}$ → see a "series" of representations.

Drinfeld/Deligne-Lusztig: G group / $\mathbb{F}_p$ ~ e.g. $GL_n\mathbb{F}_p$)

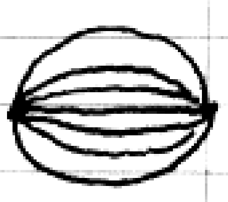
$SL_2\mathbb{F}_p \circlearrowright \{ \ell \in \mathbb{P}^1(\overline{\mathbb{F}}_p) : \mathrm{Frob}(\ell) \neq \ell \}$

$= \mathbb{P}^1(\overline{\mathbb{F}}_p) \setminus \mathbb{P}^1(\mathbb{F}_p)$

$G(\mathbb{F}_p) \circlearrowright DL_w = \{ B \in G(\overline{\mathbb{F}}_p)/B(\overline{\mathbb{F}}_p) :$
$B \underset{w}{—} \mathrm{Frob}\, B \}$

Tori: $g \in SL_2\mathbb{R}$ fall into three types:

- hyperbolic: $|\mathrm{tr}\, g| > 2$ $\longleftrightarrow$ conjugate to $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ $a \in \mathbb{R}$
  two fixed points in circle



- elliptic: $|\mathrm{tr}\, g| < 2$ conjugate to $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$
  fixed pt in interior

- parabolic: $|\mathrm{tr}\, g| = 2$ conjugate to $\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
  (& $\pm Id$)

In particular dichotomy semisimple vs unipotent (no interesting Jordan blocks in SL_2 - in general have range of intermediates).

$G \supset G^{rss}_{\subseteq G}$ regular semisimple elements:

g s.t. $Z_G(g)$ is a torus $\Longleftrightarrow$ diagonalizable over $\mathbb{C}$ with distinct eigenvalues. i.e. hyperbolic or elliptic

Hyperbolic: $Z_G(g)$ is a split torus: $Z_G(g)^\circ \simeq [(\mathbb{R}^*)^\ell]^\circ$
ℓ = rank (= dim torus over $\mathbb{C}$)

elliptic: $Z_G(g)$ is a compact torus.

General $G_{\mathbb{R}}$: have several conjugacy classes of tori; ~~each~~ [each $\simeq$ finite $\times$ $(U(1))^{n_1} \times (\mathbb{R}_+)^{n_2}$.]

$G^{rss} \subset G$ is dense always, complement real analytic submanifold. Always have split, not always have compact.

Discrete series cont.

K-types of D_n : $SU(1,1) \circlearrowright \mathbb{D}$

$$f(z) = \sum_{m\geq 0} a_m z^m \,(dz^{n/2})$$

$g: \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta}\end{pmatrix} \in K$

$$z^m dz^{n/2} \mapsto (g^{-1}z)^m (g^{-1\prime})^{n/2} (dz^{n/2})$$
$$= e^{-2mi\theta} z^m \cdot (e^{2i\theta})^{n/2} \cdot (dz^{n/2})$$
$$= e^{-(2m-n)i\theta} z^m (dz^{n/2})$$

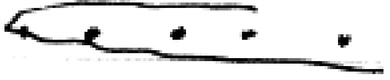
$\Rightarrow$ get all negative even steps from $-n$

: Verma module

$D_n^- = D_n^*$ contragredient $\quad g\cdot f(z) = f(gz)(g')^{n/2}$

$\longleftrightarrow \quad f(\bar z)\, d\bar z^{n/2}$

$$\bar z^m d\bar z^{n/2} \longmapsto e^{(2m+n)i\theta} \bar z^m d\bar z^{n/2}$$



Discrete series : generalization of f-d reps of compact Lie groups K : $K/T = K_{\mathbb{C}}/B_{\mathbb{C}}$,
characters of $T \Rightarrow$ hol. line bundles on K/T
take holomorphic sections.

$G_{\mathbb{R}} \supset T$ compact torus $\overset{\boxed{\text{IF } \exists}}{\Longrightarrow}$ $G_{\mathbb{R}}/T$: ~~[illegible]~~

Claim. $G_{\mathbb{R}} \circlearrowright G_{\mathbb{C}}/B_{\mathbb{C}}$ has finitely many orbits,
· each open orbit $\simeq G_{\mathbb{R}}/T \subset G_{\mathbb{C}}/B_{\mathbb{C}}$

$\rightarrow$ gets complex structure, $\mathcal{L}_\chi$ complex line bundles for χ char of T, equivariant wrt $G_{\mathbb{R}}$
$\Rightarrow$ holomorphic sections [with appropriate growth conditions] give discrete series. (Schmid)

Unitarity: Invariant (Poincaré) measure on $\mathbb{H}$: $\frac{dx\,dy}{y^2}$
$\longleftrightarrow \frac{|dz\,d\bar z|}{1-|z|^2}$ on $\mathbb{D}$

$\Rightarrow$ define $\langle f, g\rangle = \int_{\mathbb{D}} f\bar g\, (1-|z|^2)^{n-2}\, |dz\,d\bar z|$

$$= \int_{\mathbb{H}} f\bar g\, y^{n-2}\, dx\,dy$$

$f\,dz^{n/2}$
$= f(1-|z|^2)^{n/2} \frac{dz^{n/2}}{(1-|z|^2)^{n/2}}$

So honest $D_n = \{ f \text{ holomorphic on } \mathbb{H},\ \|f\|^2 < \infty \}$

Unitary representation of $G_{\mathbb{R}}$.

<u>Theorem</u> V (unitary) irrep is a direct summand of $L^2(G_{\mathbb{R}})$

$\iff$ V is a discrete series representation.

- discrete part of the spectral decomposition of $L^2(G_{\mathbb{R}})$

$\iff$ $\langle \pi(g)v, w\rangle \in L^2(G)$ for some $\neq 0$ ($\iff \forall$) v, w

matrix coefficients in L^2.

- like reps of $U(1)$, not like reps of $\mathbb{R}$.

Principal series $V_{b,s}$ unitarily induced from

$\mathbb{C}_{it} \otimes \left(\mathbb{C}_{1} = \delta_{G/B}^{\frac{1}{2}}\right)$ for $\boxed{s = it - \frac{1}{2}}$ $t \in \mathbb{R}$.

$a \mapsto |a|$

$f(x)\,|dx|^{-\frac{s}{2}}\ \bar{g}(x)\,|dx|^{-\bar{s}/2} = f(x)\bar{g}(x)\,|dx|$

$\to$ can integrate.

Honest principal series: L^2 wrt this norm.

<u>Are these all the unitary reps?</u>

Study question infinitesimally, on H-C modules.

<u>Theorem</u> If V is an irred $(\mathfrak{g}, K)$ module, $\langle , \rangle$ pos def. invariant inner product (ie $\mathfrak{g}_{\mathbb{R}}$ skew-Hermitian, $K_{\mathbb{R}}$ unitary) $\Rightarrow$ $V, \langle , \rangle$ comes from a unitary H-C module.

$\langle , \rangle \iff V \xrightarrow{\sim} \bar{V}^*$, nec. unique ($V$ irreducible)

$\langle hv, w\rangle = -\langle v, hw\rangle$ $\langle ev, w\rangle = -\langle v, ew\rangle$ $\langle fv, w\rangle = -\langle v, fw\rangle$

$H = i(e-f)$ $Y = \frac{1}{2}(h + ie + if)$ $X = \frac{1}{2}(h - ie - if)$

$\Rightarrow H = H^*$, $X^* = -Y$.

Unitarity & Unitary Induction

$G \supset H \circlearrowright (V, \langle , \rangle)$ unitary representation, how do we induce to unitary rep of G? e.g. $V = \text{triv}$.

$\text{Ind}_H^G V$ = Functions on G/H, not unitary without choice of a measure, can't integrate functions.

But can integrate densities $f\,|dz|$

so can pair $\frac{1}{2}$ densities $\langle f\,|dz|^{\frac{1}{2}}, g\,|dz|^{\frac{1}{2}} \rangle = \int f\bar{g}\,|dz|$

What are densities on G/H?

$T_1^* G/H = (\mathfrak{g}/\mathfrak{h})^*$ so ~~tensors~~ densities are sections of G-equivariant bundle associated to H-rep

$$\delta = \delta_{G/H} : h \longmapsto |\text{Det}(\text{Ad}\,h : (\mathfrak{g}/\mathfrak{h})^*)|$$

$$= |\text{Det}(\text{Ad}\,h : \mathfrak{h}^*)| \,/\, |\text{Det}(\text{Ad}\,h : \mathfrak{g})|$$

$$= \delta_G(h) / \delta_H(h)$$

In our case $G = SL_2\mathbb{R}$, $H = B_\mathbb{R}$

$\delta\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix} = |a^2| / 1$: action on $\begin{pmatrix} e \\ h \end{pmatrix}$ is $\begin{pmatrix} a^2 e \\ h \end{pmatrix}$ $\begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}$

action on $\mathfrak{sl}_2$: preserves nondegenerate form $\text{tr}(XY)$ ... $\begin{pmatrix} a^2 & & \\ & 1 & \\ & & a^{-2} \end{pmatrix}$

Note this has a well defined square root

$\delta^{\frac{1}{2}} = |a|$, $\Rightarrow$ bundle of $\frac{1}{2}$ densities on $\mathbb{RP}^1$ (or $\mathbb{CP}^1$) well defined.

$$\text{UInd}_H^G(V, \langle , \rangle) = \text{Ind}_H^G(V\delta^{\frac{1}{2}}) = \left\{ \begin{array}{l} F : G \to \mathbb{C} : \\ F(gh) = \delta^{-\frac{1}{2}}(h)\,\pi_V^{-1}(h)\,F(g) \\ \& \int_{G/H} \langle F, F \rangle < \infty \end{array} \right\}$$

Here $\langle F(gh), G(gh) \rangle = \langle \delta^{-\frac{1}{2}}(h)\,\pi_V(h)\,F(g), \delta^{-\frac{1}{2}}(h)\,\pi_V(h)\,G(g) \rangle$

$= \delta(h)^{-1} \langle F(g), G(g) \rangle$ is a density $\Rightarrow$ inner product. $\Rightarrow$ unitary rep.

(even case) $\lambda \notin \mathbb{Z}$: Pick $v_0 \in V_0$, $X v_n = v_{n+2}$ basis

$Y v_n = c_{n-2} v_{n-2}$, $\|v_n\|^2 = a_n^2$

$$X Y v_n - Y X v_n = n \cdot v_n = (c_{n-2} - c_n) v_n$$

$$\underset{a_{n+2}^2}{\overset{}{\langle X v_n, v_{n+2} \rangle}} = \underset{-\bar{c}_n a_n^2}{- \langle v_n, Y v_{n+2} \rangle} \Rightarrow c_n < 0 \text{ real}$$

$$(C+1) v_n = \lambda^2 v_n \Rightarrow \lambda^2 v_n = v_n (1 + n^2 + 2c_{n-2} + 2c_n)$$

$$\Rightarrow \lambda^2 - n^2 - 1 = 2(c_n + c_{n-2}) = 4c_n + 2n$$

$$\Rightarrow 4c_n = \lambda^2 - (n+1)^2 < 0 \quad \forall n$$

So $\lambda \in i\mathbb{R}$ OR $-1 < \lambda < 1$

$(s \in i\mathbb{R} - 1)$ $(-2 < s < 0)$

unitary principal series

complementary series

nondevious inner product

(check $\lambda = 0$ still ok) limits of discrete series $\supset \subset$

$\lambda \in \mathbb{Z}$ ok for discrete series, not f.d. reps

not L^2.

Note: Casimir invariant under $\lambda \to -\lambda$ $s \to -s-2$

SO ARE $V_{b,s}$ $s \notin \mathbb{Z}$ $\Downarrow$ intertwiners, functional eqn.

Note: • Discrete series $\subset L^2(G)$ discrete

• Principal series $s \in i\mathbb{R} - 1$ $\subset L^2(G)$ continuous spectrum: ie. no individual one appears but $\int_{i\mathbb{R}-1} V_{b,s} \subset L^2(G)$, with appropriate measure $\Rightarrow$ Plancherel theorem for $L^2(G)$

ie any $f \in L^2(SL_2\mathbb{R})$ can be written

$$f = \int_{p \in \{i\mathbb{R}-1 \,\cup\, \mathbb{Z}\}} f_p \, d\mu(p)$$

H-C c function

$$\|f\|^2 = \int \|f_p\|^2 \, d\mu(p)$$

$f \in C_c^\infty(SL_2\mathbb{R})$ $\quad Tr_{L^2(SL_2\mathbb{R})}(f) = * \sum_n \epsilon_n Tr f|_{D_n} + * \int_{i\mathbb{R}-1} c(s) \, Tr f|_{V_{b,s}} \, ds$

Principal Series Intertwiners

$s \notin \mathbb{Z}$, H-C modules for V_s & V_{-s-2} are isomorphic :;
$\mathrm{Mod}^{C+1=\lambda^2}$ has a single irreducible!
How do we see this on representations?

: realize action of Weyl group $\mathbb{Z}/2$.
s-line ⟷ characters of torus $\simeq \mathbb{R}_+$ (forget sign rep b)
$Z(U\mathfrak{g})$ give functions on $\mathbb{C} = \widehat{\mathbb{R}_+} = (\mathfrak{h}_\mathbb{C})^*$
by looking at eigenvalue on principal series reps
$(C+1)(s) = (s+1)^2$: functions are $\delta^{\frac{1}{2}}$-shifted W-invariant.
Same for Verma modules: $s \in (\mathfrak{h}_\mathbb{C})^*$ ⟹ $\mathrm{Verma}_s = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_s$
$C+1 = 4fe + 2h + h^2 + 1$, so on h.w vector
$(C+1)v_n = (n+1)^2 v_n$. (not $(\mathfrak{g},K)$-modules for $s \notin \mathbb{Z}$)
Analog of h-w vector for "unramified principal series" ($b = 1$ not sign) : K-fixed vector v_0, unique up to scalar. $(V_s)^K$ 1-dim, $C+1$ preserves this vector space & acts by scalar.

So $Z(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{h}^*]^W$ W acting in shifted way!
H-C isomorphism.
Note: • ~~$\mathbb{C}[\mathfrak{h}^*]^W \simeq \mathrm{Rep}_\mathbb{C}(SL_2\mathbb{C})^\vee$: representation ring of group with dual torus~~
– Sat~~ake isomorphism.~~
• $\mathbb{C}[\mathfrak{h}^*]^W$ ~~$\simeq (\mathcal{D}^{const}_{\mathfrak{h}})^W$~~ constant coeff diffops

• Mellin transform implicit: we're studying characters of $\mathbb{R}_+$. $(\mathbb{R}_+)^\wedge$ identified with $\mathbb{R}$ via transport of Fourier transform!
$$\hat{f}(s) = \int_0^\infty t^s f(t)\,\frac{dt}{t} \qquad (t^s = e^{s \log t})$$

– we'll see Γ functions arising naturally: $\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}$
Mellin transform of additive character,
comes from thinking of $\mathbb{R}_+ \subset \mathbb{R}^\wedge$: e.g.
action on $\mathbb{RP}^1 = \mathbb{R} \cup \infty$ ..

f is $-\frac{s}{2}$ density, $\tilde{f}$ $\frac{s}{2}$ density, to integrate take out a density $\Rightarrow$ left with $\frac{s}{2}-1=\frac{s-2}{2}$ density ie V_{-s-2}.

__Corollaries__: • Inner product on complementary series reps:
$\langle , \rangle \iff V_s \xrightarrow{\sim} \bar{V}_s^{*} \simeq V_{-s-2}$ via integration pairing $f_1 |dx|^{-\frac{s}{2}} \cdot f_2 |dx|^{-\frac{(-s-2)}{2}} = f_1 f_2 |dx|$.

So get such an isomorphism from M, need to check pos definite $\iff -2 \leq s < 0$.

• Projection of principal series V_n to discrete series D_n: $D_n = \mathrm{Im}(M : V_n \longrightarrow V_{-n-2})$

$\rightsquigarrow$ poles at negative ~~even~~ integers ...

Will give functional equation of Eisenstein series & L, ζ functions!

Intertwiners $T = \left\{\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\right\}$. $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ acts by $a \mapsto a^{-1}$ on $T_{\mathbb{R}}$ & acts on Borel's containing T: only two such, ways to order $\mathbb{C}^2$ into eigenspaces:

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} * & * \\ & * \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \begin{pmatrix} * & \\ * & * \end{pmatrix}$$

So get an obvious map $\mathrm{Ind}_{B_{\mathbb{R}}}^{G_{\mathbb{R}}} \mathbb{C}_s \longrightarrow \mathrm{Ind}_{\bar{B}_{\mathbb{R}}}^{G_{\mathbb{R}}} \mathbb{C}_{-s}$:

$f \longmapsto \tilde{f}(g) = f(gw)$. $\quad f\left(g\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix}\right) = |a|^s f(g)$

$\Rightarrow \tilde{f}\left(g\begin{pmatrix} a & \\ b & a^{-1}\end{pmatrix}\right) = f\left(g\begin{pmatrix} a & \\ b & a^{-1}\end{pmatrix}w\right) = f\left(gw\begin{pmatrix} a^{-1} & b^{-1} \\ & a\end{pmatrix}\right) = |a|^{-s}\tilde{f}(g)$.

Now want to go back to our principal series, ie need right N invariance $\Rightarrow$ impose it:

$$M(f) = \int_N \tilde{f}(gn)\,dn = \int_N f(gnw)\,dn$$

$\left[dn = dx \quad \begin{pmatrix} 1 & x \\ & 1\end{pmatrix} = n.\right]$

Check $(M(g f))(h) = \int_N f(g^{-1} h n w)\,dn = \int_N f(g^{-1}(hnw))\,dn$
$= (g\cdot M(f))(h)$ intertwines action.

$M(f)\left(g\cdot\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix}\right) = \int_N f\left(g\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & b+n \\ & 1\end{pmatrix}w\right)dn$

$= \int_N f\left(g\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & n \\ & 1\end{pmatrix}w\right)dn = \int_N f\left(g\begin{pmatrix} 1 & a^2 n \\ & 1\end{pmatrix}\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}w\right)dn$

$n' = a^2 n$
$dn' = a^2 dn$

$= \int_N f\left(g\begin{pmatrix} 1 & n' \\ & 1\end{pmatrix}w\begin{pmatrix} a^{-1} & \\ & a\end{pmatrix}\right)\frac{dn'}{a^2}$

$= |a|^{-s-2}\, M(f)(g)$.

- Claim: converges for $s > 1$, analytic continuation with poles.

Geometrically:

$$\begin{tikzcd} & \mathcal{Fl}_w = \{\ell \neq \ell'\} \arrow[dl] \arrow[dr] & \\ \mathbb{RP}^1 & \{N\cdot\ell\} \arrow[d] & \mathbb{RP}^1 \\ & \ell' & \ell \end{tikzcd}$$

: $G \circlearrowright G/B \times G/B \longleftrightarrow W$

$M(f) = \int_{\pi_2} \pi_1^*(f)$: $M(f)(\ell') = \int_{\ell\neq\ell'} f(\ell)\,d\ell$

[illegible]