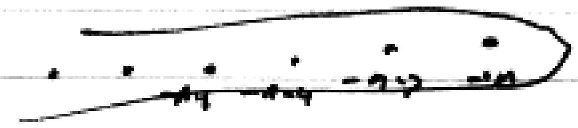
Character theory

Character of discrete series: 

$\leftrightarrow$ Verma module. Can assign Lie algebra character / K character
remember just keep track of $\displaystyle\sum_{\text{Weight spaces } \lambda} q^\lambda = \sum_\lambda e^{i\lambda x}$

In our case (Verma module) $\Rightarrow$ $\dfrac{e^{-inx}}{1-e^{-2ix}} = \dfrac{e^{(1-n)ix}}{e^{ix}-e^{-ix}} = \dfrac{q^{-n+1}}{q-q^{-1}}$

For principal series, K-character is $\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$

$\Theta^K_{V_{\lambda,\varepsilon}}$ $\displaystyle\sum_{-\infty}^{\infty} q^n = \sum_{-\infty}^{\infty} e^{inx}$. This makes sense not just as formal sum but as distribution on K (Same as $L^2(S^1)$!)

$\Theta^K(f) = \sum \int f e^{inx} = f(1)$: Fourier transform of sum of all representations! Plancherel for S^1.

i.e. $\Theta^K|_{K^{rss}} = 0$: vanishes off $1, -1$.

Recall that character is a class function (when defined)
... formula $\sum q$'s is restricting class fn on compact group to a torus.
But on $G_{\mathbb{R}}$ not every element conjugate into K! just elliptic elements! ($\neq \pm 1$).

Harish-Chandra characters V admissible rep with infinitesimal character
... i.e. Z $(= \mathbb{C}[C])$ acts by scalar (λ^2+1)
$\Rightarrow$ • $f \in C_c^\infty$, $\pi(f)$ is trace class

• So $\Theta_\pi : f \longmapsto \operatorname{Tr} \pi(f)$ is a distribution, distributional character.

• Θ_π is conjugation invariant.

• $Z \subset \mathcal{D}_{G_{\mathbb{R}}}^{G_{\mathbb{R}} \times G_{\mathbb{R}}}$ bi-invariant differential operators

• (V,π) has inf. character $\lambda : Z \longrightarrow \mathbb{C}$
$\Longrightarrow$ Θ_π is joint eigenfunction of all these differential operators:

$$z \cdot \Theta_\pi = \lambda(z)\, \Theta_\pi$$

$$\pi(z \cdot f)(v) = \pi(f)(\pi(z)\cdot v) = \lambda(z)\, \pi(f)\cdot v.$$

So Θ solves large (<u>holonomic</u>) system of differential equations

Harish-Chandra describes Z via <u>radial parts</u>: [(close-Ma[illegible] system etc.)]
restriction to tori (in various conjugacy class)
... complexly $Z \simeq \mathbb{C}[\mathfrak{h}]^W = (\mathrm{Sym}\, \mathfrak{h}^*)^W \subset$ Constant coeff diff ops...
Eigensystem elliptic (transverse to $G_{\mathbb{R}}$-orbits) on G^{rss}
$\Longrightarrow$ Θ_π <u>real analytic</u> function on G^{rss}.
• globally representable by L^1_{loc} function — so determined by restriction to G^{rss}.

(characteristic directions / wave front set $\subset T^* G^{rss}$:
Lagrangian $\left\{ (g \in G_{\mathbb{R}},\ \xi \in \mathfrak{g}^* \text{ nilpotent}) : g\cdot\xi = \xi \right\}$
... determined by geometry of holonomic Z-system)

<u>At the end of the day</u> $\Delta \cdot \Theta_\pi$ determined by a collection of finite Fourier series on each representative of class of tori.
.... analog of Weyl integration formula:
integrating function on $G \longleftarrow G/T \times T$
Jacobian $= \Delta = \prod (q_i - q_j)$ — describe characters as W-symmetric finite Fourier series on T
$\longleftrightarrow$ Fourier transforms of integration on conjugacy class:

$$\Theta_{m_1 \dots m_n}(f) = \frac{1}{n!} \int_T \Delta \bar{\Delta}\, \Theta(t) \int_G f(xtx^{-1})\, dx$$

$$\left[\approx \frac{1}{n!} \sum_W (-1)^w \int_T e^{i\sum m_i t_i} \int_G f(xtx^{-1})\, dx \right]$$

$\boxed{\text{Fourier transform of orbital integrals}}$ $= \int_T e^{i\sum m_i t_i} (-1)^{n(n-1)/2} (t_1 \cdots t_n)^{-(n-1)} \Delta(t) \int_G f(xtx^{-1})\, dx$

$T < G_{\mathbb{R}}$ torus $\quad \Theta^T_\pi = \sum_{w \in W_{\mathbb{C}}} \frac{n_w \in \mathbb{Z} \cdot e^{iw(\lambda)}}{\Delta}$ $\quad \lambda \leftrightarrow$ central character

on T^{reg} roughly

for some n_w ....

Invariant under $W_{\mathbb{R}}$: (case of $K_{\mathbb{C}} = \mathbb{C}^*$ no Weyl group...

Principal series :

$V_{+,s}$: $a = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}$

$$\Theta_{+,s}(a) = \frac{e^{-st} + e^{st}}{|e^t - e^{-t}|} = \frac{e^{-st} + e^{st}}{e^t - e^{-t}} \quad \text{for } a > 1 \ (t > 0)$$

$$= e^{(s-1)t} + e^{(s-3)t} + e^{(s-5)t} + \dots + e^{(-s-1)t} + e^{(-s-3)t} + \dots$$

Note when $s = n \in \mathbb{Z}$ get

$$e^{(n-1)t} + \dots + e^{-(n+1)t} + 2\left(e^{(-n-1)t} + e^{(-n-3)t} + \dots\right)$$

[$\rightsquigarrow$ character of $D_n^\pm$ is $\frac{e^{-nt}}{e^t - e^{-t}}$ on split torus

"Same formula" as on K ...]

Note : a is not diagonalizable on $V_{+,s}$ with the above discrete eigenvalues ! this is a regularization (a action : rescaling on $\mathbb{R}$ $\leftrightarrow$ $h = x\frac{d}{dx} - s$, continuous eigenspaces)

Plancherel theorem : describe $L^2(G_{\mathbb{R}})$ in terms of $L^2(T)^W$ with appropriate measure : rep theory of $G_{\mathbb{R}}$ breaks up into that of tori !

Harmonic analysis on $\Gamma\backslash G$:

$\Gamma \subset SL_2\mathbb{R}$ discrete subgroups. Two sources of examples :

- Riemann Surfaces X compact $g>1$,

$X = \Gamma\backslash\mathbb{H} = \Gamma\backslash SL_2\mathbb{R}/K$ $\quad \Gamma : \pi_1(x) \to SL_2\mathbb{R}$

$\uparrow$

$\tilde{X} = \Gamma\backslash SL_2\mathbb{R}$ unit tangent bundle, compact.

- Arithmetic groups $\Gamma = SL_2\mathbb{Z}$ or $\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} : c \equiv 0 (N)\right\}$

$\mathcal{H} = L^2(\Gamma\backslash G)$ Unitary representation of $SL_2\mathbb{R}$ - falls into three pieces :

- discrete series $D^{\pm}(n) \subset \mathcal{H}$
- principal, complementary $\subset \mathcal{H}$ discretely
- " " $\subset \mathcal{H}$ continuously

Change notation: $\Delta = -\frac{1}{4}C = -\frac{1}{4}(H^2 + 2XY + 2YX)$.

e'vals of Δ on V_s : $C+1 = (s+1)^2 = s^2+2s+1$

$\Delta = -\frac{1}{4}(s^2+2s) = \frac{s}{2}(-1 - \frac{s}{2})$

Complementary series: $-2 \leq s \leq 0 \Rightarrow 0 \leq \Delta \leq \frac{1}{4}$ ($s=-2,0$; $s=-1$)

Principal series: $s \in -1 + i\mathbb{R}$ $\quad \frac{1}{4} \leq \Delta$ real ($s=-1$)

Discrete series $s \in \mathbb{Z} - \{-1\}$ Δ negative real

Δ = Laplace-Beltrami operator : write coords on $SL_2\mathbb{R}$

$$g = \begin{pmatrix} y^{\frac{1}{2}} & y^{-\frac{1}{2}}x \\ & y^{-\frac{1}{2}}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix} \quad \left[g\cdot i = x+iy \in \mathbb{H}\right]$$

$\Rightarrow$ calculate $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial^2}{\partial x\,\partial\theta}$

On K-invariant functions (ie fns on $\mathbb{H}$) this is usual Laplacian for hyperbolic metric.

Actually should study $L^2(\Gamma\backslash G, \chi)$
where χ is a character of Γ, assume $-I \in \Gamma$
$\Rightarrow \quad \chi(-1) = (-1)^{\varepsilon} \quad \varepsilon = 0$ or 1 :
<u>even</u> or <u>odd</u>

Reps occurring in $L^2(\Gamma\backslash G,\chi)$ are also <u>even or odd</u>.
$\Leftrightarrow$ $-I$ acts by ± 1 $\Leftrightarrow$ odd or even K-types.

<u>Note</u> $\nexists$ complementary series for $\varepsilon = 1$:
analysis of unitarity showed us $\lambda^2 < 1$ before,
in general, but with sign set $\lambda^2 - \varepsilon^2 + 2\varepsilon < 1$
$\Rightarrow \lambda^2 < 0 \Rightarrow$ principal series only.

So on $L^2(\Gamma\backslash G, \chi)$ Δ has spectrum:
- $\varepsilon = 0$ $\quad \Delta \geq 0$ or $\Delta = -\frac{s}{2}(1+\frac{s}{2})$ $\quad s \in \mathbb{Z}$
- $\varepsilon = 1$ $\quad \Delta \geq \frac{1}{4}$ or "

$0 \leq \Delta \leq \frac{1}{4}$: <u>exceptional</u> eigenvalues, complementary series.

How to detect reps : $V_{\mu,s}$ in unitary range has
unique/scalar spherical vector $\mathbb{C}V_0 = V_{1,s}^K$

So to detect V_s look for K-invariant functions in $\mathcal{H}$

i.e. functions in $L^2(\Gamma\backslash G/K)$

<u>Prop</u> The multiplicity of V_s in $\mathcal{H}$ is the dimension $(\chi = 1)$
of the $-\frac{s}{2}(1+\frac{s}{2})$ eigenspace of the Laplacian
in $L^2(\Gamma\backslash G/K) = L^2(\Gamma\backslash\mathbb{H})$ $\quad (s \notin \mathbb{Z})$.

Similarly for $\varepsilon = 1$: $V_{b,s}$ in $\mathcal{H}$ $\Leftrightarrow$
functions in $L^2(\Gamma\backslash\mathbb{H}, \chi)$
$=$ functions f on $\mathbb{H}$ with $f(\gamma x) = \chi(\gamma) f(x)$.

Really nothing special about K-invariants: $V_{\varepsilon,s}$ has 1-dim
k-eigenspace for every $k \equiv \varepsilon \ (2)$.

⇒ Same statement holds for such k:
multiplicity of $V_{\varepsilon,s}$ in $\mathcal{H}$ = dimension of
$L^2(\Gamma\backslash SL_2\mathbb{R}/K, \cancel{\mathbb{Z}}, k)$:

$$L^2(\Gamma\backslash G, \chi) = \bigoplus_{k\in\mathbb{Z}} L^2(\Gamma\backslash G/K, \chi, k)$$

$k_\theta = \begin{pmatrix}\cos & \sin\\ -\sin & \cos\end{pmatrix}$ acts by $e^{ik\theta}$

<u>Proposition</u> $L^2(\Gamma\backslash G/K, \chi, k) \cong \{ f \in L^2(\mathbb{H}) : \boxed{f(z) = \chi(\gamma)\frac{1}{(cz+d)^k} f\left(\frac{az+b}{cz+d}\right)} \}$
(F)

$\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ $L^2_\chi(\mathbb{H}, k)$

via $F(g) = (g\cdot_k f)(i) = \frac{1}{(ci+d)^k} f\left(\frac{ai+b}{ci+d}\right)$

i.e. k-invariance under Γ is same as invariance
as $\frac{k}{2}$-form ⟺ invariance under shifted action
of K. $g\cdot_k f = g\cdot(f\, dz^{k/2})$

How to detect discrete series?
no spherical vector, but have highest weight vector:

D_n

$-n-4 \quad -n-2 \quad -n \quad -n+2$ $\quad s = n-2 \qquad D_n^- \qquad n-2 \quad n \quad n+2 \quad n+4 \quad \cdots$

$Xv_{-n} = 0 \qquad\qquad Yv_n = 0$

$(C+1)v_{\pm n} = (n-1)^2 \qquad \Delta = -\frac{s}{2}\left(1+\frac{s}{2}\right) = -\frac{n-2}{2}\left(\frac{n}{2}\right)$
$= \frac{n}{2}\left(1-\frac{n}{2}\right)$

⇒ Multiplicity of D^- in $\mathcal{H}$ is
dimension of $\frac{n}{2}(1-\frac{n}{2})$ Δ eigenspace in $L^2(\Gamma\backslash G/K, \chi, n)$
D^+ " " " $L^2(\Gamma\backslash G/K, \chi, -n)$

These two spaces exchanged by complex conjugation
⇒ both multiplicities equal.

Suppose $F \leftrightarrow v_n$ vector in $L^2(\Gamma\backslash G, \chi, n)$ of Δ-eigenvalue $\frac{n}{2}(1-\frac{n}{2})$.

We know f lowest weight, $\underline{Y\cdot f = 0}$.

Coordinate calculation:

$$Y = e^{2i\theta}\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial\theta}\right)$$

Translate action to $L^2_\Gamma(\mathbb{H}, n)$ ie to k-action of $SL_2\mathbb{R}$

$$f \longmapsto \frac{1}{(cz+d)^n} f\left(\frac{az+b}{cz+d}\right) :$$

$$Y_n = -iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{n}{2} = -(z-\bar{z})\frac{\partial}{\partial \bar{z}} - \frac{n}{2}$$

$$= -2iy\frac{\partial}{\partial \bar{z}} - \frac{n}{2}$$

$$\left[\begin{matrix} z = x+iy \\ \bar{z} = x-iy \end{matrix} \qquad -\frac{\partial}{\partial \bar{z}} = -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]$$

$$Y_n \cdot f = 0 \iff \frac{\partial}{\partial \bar{z}} \cdot (y^{-n/2} f) = 0$$

So $f y^{-n/2}$ is holomorphic

ie $f\left(\frac{dz}{y}\right)^{n/2}$ holomorphic Γ-int $\frac{n}{2}$-form on $\mathbb{H}$.

[recall $d\mathrm{vol}_{\mathbb{H}} = \frac{|dz\, d\bar{z}|}{2y^2} = \frac{dx\,dy}{y^2}$]

So: principal, complementary series in $\Gamma\backslash \mathbb{H}$ characterized by eigenfunctions of Δ, discrete series in forms on $\Gamma\backslash\mathbb{H}$ characterized by holomorphy.

<u>Theorem</u> Γ cocompact $\Rightarrow \mathcal{H} = L^2(\Gamma\backslash G, \chi)$ decomposes into a Hilbert space direct sum of irreducibles,

Multiplicity of V_s s.t. $\uparrow$ principal or complementary

is dim of $-\frac{3}{2}(1+\frac{s}{2})$ Δ-eigenspace in $L^2(\Gamma\backslash\mathbb{H}, \chi)$

Multiplicity of $D^{\pm}(n)$ is dim of holomorphic $\frac{n}{2}$-forms on $(\Gamma\backslash\mathbb{H}, \chi)$

(

Classical modular forms

$\Gamma = SL_2\mathbb{Z}$ or $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \ (N) \right\}$

$\Gamma \backslash SL_2\mathbb{R}$ = unimodular lattices in $\mathbb{R}^2$ ($SL_2\mathbb{Z} = \mathrm{Stab}(\mathbb{Z}\oplus\mathbb{Z})$)
or lattices Λ with a line in Λ (mod N) distinguished.
($\mathbb{Z}$-sublattice)

$\Gamma \subset SL_2\mathbb{Q}$. ~~If~~ $c \in \mathbb{Q}\cup\infty \subset \mathbb{RP}^1$ ~~then~~ $\Longleftrightarrow$
Stab c contains nontrivial unipotent elements
eg Stab ∞ $\cap$ unipotents = translations in Γ

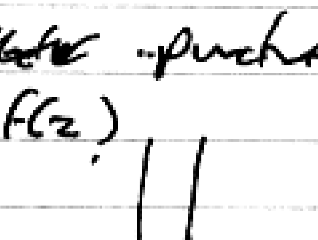
Cusps of Γ: Γ orbits on $\mathbb{Q}\cup\infty = SL_2\mathbb{Q}/B_{\mathbb{Q}}$.
$= SL_2\mathbb{Z}/B_{\mathbb{Z}}$.
$SL_2\mathbb{Z}$ acts transitively, one cusp.

$\Gamma\backslash\mathbb{H} = Y_0(N)$ $\qquad \Gamma\backslash\overline{\mathbb{H}} = \Gamma\backslash(\mathbb{H}\cup\mathbb{QP}^1) = Y_0(N)$

$SL_2\mathbb{Z}$ gen. by $z\mapsto z+1$, $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ $\qquad$ All finite volume.
& $z \mapsto -\frac{1}{z}$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Def. A holomorphic modular form for Γ is a hol function f on $\mathbb{H}$ s.t. • $f(\gamma z) = (cz+d)^k f(z)$ $\quad k$ = weight
• f holomorphic at every cusp of Γ.

Holomorphy at cusps: Cusp looks like ~~cylinder~~ punctured unit disc:
$SL_2\mathbb{Z}$: $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \in \Gamma \Rightarrow f(z+1) = f(z)$
$z \mapsto q = e^{2\pi i z}$ sends strip



to

punctured unit disc, $i\infty \mapsto 0$.
f has Laurent expansion in q (q-expansion)

$$f(q) = \sum_n a_n q^n = \sum a_n e^{2\pi i n z}$$

Holomorphy: $a_n = 0$ if $n < 0$.
Cusp form: $a_0 = 0$: vanishes at cusp

Same for $\Gamma_0(N)$: demand holomorphy (/cuspidality) at every cusp.

~~Note.~~ f cuspidal $\Leftrightarrow$ f holomorphic & rapidly decreasing:

$$|f(x+iy)| \le C_t\, y^{-t} \quad \forall t \quad \text{as } y \to \infty$$
with x bounded

... follows from $|e^{2\pi i k z}| = e^{-2\pi k y}$

[holomorphic at ∞ : f of moderate growth:
$|f(x+iy)| \le C \cdot y^n$ some n $\quad y \to \infty$, x bdd]

Cusp form $\Rightarrow$ ~~$f \in L^2$~~
$$F(g) = f(g_{\cdot}\, i) \quad \text{is in } L^2(\Gamma \backslash SL_2\mathbb{Z})$$
... f rapidly decreasing at cusps
$\Rightarrow$ F bounded on $\Gamma \backslash$ ~~SL_2~~ $SL_2\mathbb{R}$
$\Longleftrightarrow$ $f\, dz^{n/2}$ bounded $n/2$ form on $\Gamma \backslash \mathbb{H}$.

So cusp modular forms $\longleftrightarrow$ appearances of $D(n)$ in $L^2(\Gamma \backslash SL_2\mathbb{R})$. Cusp forms $S_k(N) \subset M_k(N)$ modular finite dim

<u>Maass forms</u>: f smooth Γ-invariant function on $\mathbb{H}$ of moderate growth at cusps & Δ-eigenfunction.
($\Rightarrow$ real analytic)

Maass cusp form: rapidly decreasing at cusps

$\longleftrightarrow$ appearances of V_s in $L^2(\Gamma \backslash SL_2\mathbb{R})$.

Maass forms don't have holomorphic Fourier expansions but still $f(z+1) = f(z)$

$$\Rightarrow f(x+iy) = \sum_{-\infty}^{\infty} a_n \left[\begin{matrix}\text{fn of } y, s \\ \text{which is } \Delta\text{-eigenfn} \\ \text{e'val } -\tfrac{s}{2}(1-\tfrac{s}{2})\end{matrix} \right] e^{2\pi i n x} \Big]$$

... turns out need ~~Bessel~~ Bessel function
$$\sqrt{y}\; K_{\nu}(2\pi |n| y) \qquad \Delta = \tfrac{1}{4} - \nu^2$$

$$K_\nu(y) = \frac{1}{2} \int_0^\infty e^{-y(t+t^{-1})/2}\, t^{\nu}\, \frac{dt}{t}$$

$$\left\{ y^2 \frac{d^2}{dy^2} + y \frac{d}{dy} - (y^2 + \nu^2) \right\} K_\nu(y) = 0$$

L-functions f modular or Maass form $\to$
can "decompose" f under action of torus
$\mathbb{R}_+$ associated to cusp at ∞:
Stab ∞ = $B_{\mathbb{R}} \supset \mathbb{R}_+$ rescaling y, $t: y \mapsto ty$.

$\mathbb{R}_+ \simeq \mathbb{R}$ so representations of $\mathbb{R}_+$ described by
a Fourier transform, $(\mathbb{R}_+)^\wedge = \mathbb{R}_s$ Same s-line as in principal series rep.

$$\hat{f}(s) = \int_0^\infty f(t)\, t^s \frac{dt}{t}$$ Mellin transform

$$\left(t^s = e^{s \log t},\ \frac{dt}{t} = d(\log t) \right).$$

f cusp form
$$\int_0^\infty f(iy)\, y^s \frac{dy}{y} = \int_0^\infty \sum_1^\infty a_n(f)\, e^{-2\pi n y} y^s \frac{dy}{y}$$

$$= \sum_0^\infty a_n \int_0^\infty e^{-t} (2\pi n)^{-s} t^s \frac{dt}{t} \qquad (t = 2\pi n y)$$

$$\Lambda(f,s) := = (2\pi)^{-s} \Gamma(s) \sum_0^\infty \frac{a_n}{n^s} \qquad \Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}$$ Gamma function

— Mellin transform of additive character of $\mathbb{R}_+$ — compatibility between $+$ & $\cdot$ on $\mathbb{R}$.

$$L(f,s) = \sum_1^\infty \frac{a_n}{n^s}$$ L-function

Maass case! $e^{-t} \rightsquigarrow K_\nu(t)$
$$\int_0^\infty K_\nu(y)\, y^s \frac{dy}{y}$$: two Mellin transforms multiplied,
$$= 2^{s-2}\, \Gamma\left(\frac{s+\nu}{2}\right) \Gamma\left(\frac{s-\nu}{2}\right)$$

$$\Lambda(f,s) = \int_0^\infty f(iy)\, y^s \frac{dy}{y} = \pi^{-s}\, \Gamma\left(\frac{s+\nu}{2}\right) \Gamma\left(\frac{s-\nu}{2}\right) L(f,s)$$

$$L(f,s) = \sum \frac{a_n}{n^s}.$$

$\Lambda(f,s)$ converges for all s:

- $f(iy) \to 0$ exponentially as $y \to \infty$
- $f(iy) = (-1)^{k/2} y^{-k} f(i/y)$ $\qquad w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \in \Gamma$

So $f(iy) \longrightarrow 0$ faster than polynomially as $y \to 0$.

$\Rightarrow$ $L(f,s)$ has analytic continuation to an entire fn on $\mathbb{C}$ since Γ entire, nonzero.

Functional equation: $\Lambda(f,s) = (-1)^{k/2} \Lambda(k-s, f)$

reflection in $s=0$ or $s' = s-1 = 1$

(Mellin : $\int f(t)\, t^{s-1}\, dt$)

$$\int_0^{\infty} f(iy)\, y^s \frac{dy}{y} = \int_0^{\infty} (-1)^{k/2} y^{-k} f(i/y)\, y^s \frac{dy}{y}$$

$$= (-1)^{k/2} \int_0^{\infty} f(it)\, t^{k-s} \frac{dy}{y} \qquad t = \frac{1}{y}$$

i.e. functional equation comes from Weyl group action!

Def An automorphic form is a function $f \in C^\infty(\Gamma \backslash SL_2\mathbb{R})$
s.t.
- f is K-finite ($\Rightarrow$ lives in assoc. $(\mathfrak{g},K)$-module)
- f is Z-finite
- f has moderate growth (tempered)

Z-finite: $Z\cdot f$ finite dimensional, i.e. f annihilated by finite codim ideal of Z
$\Leftrightarrow$ lives in span of fin many generalized eigenspaces of C

Fix I
A_I admissible $(\mathfrak{g},K)$-module

Theorem: For any $I \subset Z$ fin codimension, $\sigma \subset \mathbb{Z}$ finite subset $\Rightarrow$ $A_{\sigma,I} = \{ f \text{ automorphic},\ If = 0,\ f \in (C^\infty)_\sigma \}$ is finite dimensional.

Tempered: $|f(g)| \le C y(g)^N$ $\quad y(g)$: $\begin{pmatrix} y^{1/2} & \\ & y^{-1/2}\end{pmatrix}$-component
for all g with $y(g) >$ some constant.

Eisenstein Series & Constant Terms

$$\begin{tikzcd} & \Gamma_\infty\backslash G \arrow[dl] \arrow[dr] & \\ \Gamma\backslash G & & N\Gamma_\infty\backslash G \end{tikzcd}$$

$\Gamma_\infty = \Gamma \cap B = \begin{pmatrix} a & n \\ 0 & a^{-1}\end{pmatrix}$ $(a, b \in \mathbb{Z}) = \begin{pmatrix} \pm & n \\ & \pm \end{pmatrix}$

$N = \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}$

$N\Gamma_\infty\backslash G = \pm\backslash \mathbb{R}^2 \setminus 0$.

$$A(\Gamma\backslash G) \underset{C}{\overset{E}{\rightleftarrows}} A(N\Gamma_\infty\backslash G)$$

Constant term: $C(f)(g) = \int_{\Gamma_\infty\backslash N} f(n\cdot g)\, dn$

Takes Γ-invariant function to $N\Gamma_\infty$-invariant function.
(Γ_∞ normalizes N).
$\Gamma\backslash N \cong S^1$: $\quad$ $-\frac{1}{2}$ $\frac{1}{2}$ Ng

$C(\text{modular form } \sum a_n q^n) = a_0$ — $a_m = \int_{\Gamma\backslash N} f(ng) e^{-inm}\, dn$
zeroth Fourier coefficient

Constant term of Eisenstein series:

$$\Gamma_\infty \backslash \Gamma = \mathbb{P}^1_{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cup \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \right\}$$

$\begin{pmatrix} \cancel{\pm} & s \\ 0 & \pm \end{pmatrix}$

$$C E(f)(g) = \int_{\Gamma \backslash N} E\left(f\left(\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} g\right)\right) dn$$

$$= \int_{\Gamma \backslash N} \sum_{x \in \mathbb{Z}} f\left(w_0 \begin{pmatrix} 1 & n+x \\ & 1 \end{pmatrix} g\right) dn + f(g)$$

$$= \int_N f\left(w_0 \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} g\right) dn + f(g)$$

$$= M_s(f)(g) + f(g)$$

ie, get f plus the intertwining operator $M_s(f)$

One proves M, E have analytic continuation simultaneously!

Corollary Functional eqn for Eisenstein series:

$$E(f, s) = E(M_s f, -s)$$

– both sides have same constant term $\Rightarrow$ difference is cusp form. But Eisenstein series $\perp$ cusp forms:

$$\int_{SL_2\mathbb{Z} \backslash SL_2\mathbb{R}} E(f, g)\phi(g)\, dg =$$

$$\int_{\Gamma \backslash G} \left(\sum_{\Gamma_\infty \backslash \Gamma} f(\gamma g)\right) \cdot \phi(g)\, dg = \int_{\Gamma_\infty \backslash G} f(g)\phi(g)\, dg$$

$$= \int_{N\Gamma_\infty \backslash G} f(g) \left(\int_{\Gamma_N \backslash N} \phi(ng)\, dn\right) dg = 0.$$

Calculating Eisenstein series: $\qquad \gamma g = n y k$.

$$E(f,s) = \sum_{\Gamma_\infty\backslash\Gamma} f_s(\gamma g) = \sum_{\Gamma_\infty\backslash\Gamma} y(\gamma g)^{\frac{s}{2}} f(k(\gamma g))$$

$\Gamma_\infty\backslash\Gamma \longleftrightarrow (c,d)=1 \quad (c,d)\in\mathbb{Z}^2 \qquad g(i) = z = x+iy$

$$E(1,s) = \sum_{(c,d)=1} \frac{y^{s/2}}{|cz+d|^s} \qquad \left[\mathrm{Im}(\gamma.z) = \frac{\mathrm{Im}\, z}{|cz+d|^2}\right]$$

Summing over $(c,d)=1$ & over $n\in\mathbb{Z}_+$

Summing over all $c,d \in \mathbb{Z}^2-0$:

$$\sum_{\mathbb{Z}^2-0} \frac{y^{s/2}}{|cz+d|^s} = \sum_{\mathbb{Z}\ni n>0}\ \sum_{(\mathbb{Z}^2)_{prim}} \frac{y^{s/2}}{|cz+d|^s}\cdot\frac{1}{n^s}$$

$$= \zeta(s)\, E(f,s)$$

~~Replace 1 by character func of K~~

$\cancel{\Rightarrow E(f_k,s) \Leftrightarrow \sum_{(c,d)=1} \frac{y^{s/2}}{|cz+d|^s}\cdot\frac{1}{(cz+d)^k}}$ ~~make fn on $\Gamma\backslash SL_2\mathbb{R} \to \Gamma\backslash\mathbb{H}$~~

$\cancel{\zeta(s) E(f_k,s) = \sum_{\mathbb{Z}^2-0}}$

Now replace 1 by character χ_k of $K \rightsquigarrow$ function $\chi_{k,s}$ on $N^{SL_2\mathbb{R}}$ in V_s representation

$E(\chi_{k,s},s)(g)$ on $\Gamma\backslash SL_2\mathbb{R}$ $\quad$ K character : k-form on $\Gamma\backslash\mathbb{H}$

$$E(\chi_{k,s},s)(g) = \sum_{(c,d)=1} \frac{1}{|cz+d|^s}\ \frac{1}{(cz+d)^k} \cdot y^{s/2}$$

Used to analytically continue holomorphic Eisenstein series

$$\zeta(k)\, E(\chi_k, s=0) = \sum_{\mathbb{Z}^2-0} \frac{1}{(cz+d)^k}\,.$$

p-adic numbers

Geometry of $\mathbb{Z}$: think of integer as function on prime numbers. $n : [p] \mapsto n \bmod p \in \mathbb{Z}/p\mathbb{Z}$

$\mathbb{Z} \ni p$ is coordinate near $[p]$: generates $\{$functions vanishing at $[p]\} = p\mathbb{Z}$ max ideal.

Write Taylor series for functions around p:

$n = a_0 + a_1 p + a_2 p^2 + \dots + a_k p^k \quad a_i \in \mathbb{Z}$ not divisible by p

to get unique ask $0 \le a_i < p$

$a_0 \equiv n \ (p)$. only meaningful coefficient is leading term : $n = bp^m \quad p \nmid b \quad m =$ order of vanishing

Set $|n|_p = \frac{1}{p^m} \qquad |mn|_p = |m|_p |n|_p$

$|n| < 1 \iff$ ~~pt~~ n vanishes at p.

$\mathbb{Z}_p =$ formal Taylor series in p,

$x = a_0 + a_1 p + \dots \qquad 0 \le a_i < p$

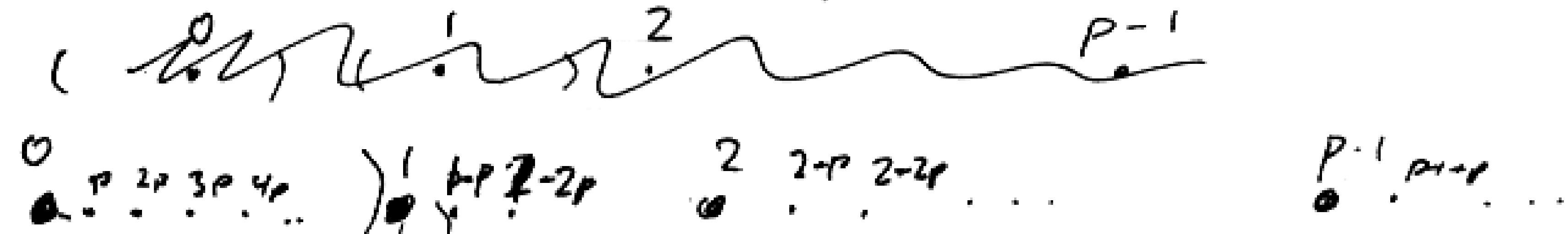
with addition & product what they have to be:

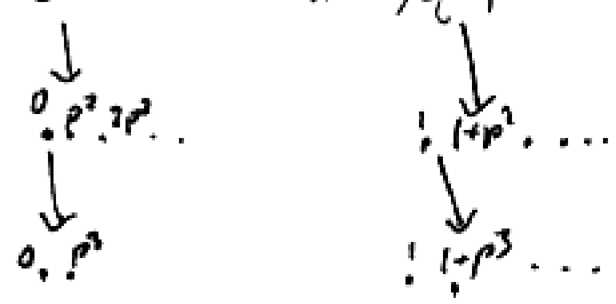
$\mathbb{Z}_p \to \dots \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^2 \to \mathbb{Z}/p$

inverse limit : give k-jet for every k compatibly.

$\mathbb{Z}_p =$ completion of $\mathbb{Z}$ wrt norm $|x|_p$: add lower & lower terms, in geometric series

$|x+y|_p \le \max\{|x|_p, |y|_p\}$ order of vanishing.





Cantor set : $\mathbb{Z}_p$ compact

$\mathbb{Z}$ totally bounded

$\mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p/p^n\mathbb{Z}_p$.

Iwasawa decomposition: $G = BK = \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \cdot SL_2(\mathbb{Z}_p)$.

$\Rightarrow$ V_s determined by its restriction to K:

$(V_s)^{K_i} \leq [K : K_i]$. <u>admissible</u>: all finite (smooth $\Rightarrow$ admissible)

Spherical vector: $V_s \supset (V_s)^{SL_2\mathbb{Z}_p}$:

take function 1 on K, invariant under $K \cap B = \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}$ $a, b \in \mathbb{Z}_p$. extends uniquely to function on G in V_s.

- analog of <u>even</u> V_s in ~~R~~ Real case.

e.g. if $n \not\equiv 0\ (p) \Rightarrow \frac{1}{n} \in \mathbb{Z}_p$:
just like Taylor series, solve $(a_0 + a_1 p + a_2 p^2 + \dots) n = 1$
$a_0 \equiv \frac{1}{n}\ (p)$ etc

$\mathbb{Q}$ give Laurent series at p : ratio of Taylor series
$x = \frac{a}{b} p^m$, $p \nmid a, b$ $\Rightarrow$ pole of order m.

$\mathbb{Q}_p$ = field of fractions of $\mathbb{Z}_p$ = Laurent series $\sum_{i \geq -N}^{\infty} a_i p^i$
$= \bigcup_{i>0} p^{-i} \mathbb{Z}_p$
$|x|_p = \frac{1}{p^m}$
$|x|_p \leq 1 \iff x \in \mathbb{Z}_p$: unit ball
$\mathbb{Z}_p$ = max compact subgroup of $\mathbb{Q}_p$. $p\mathbb{Z}_p$ open unit ball.

$\boxed{\text{NOTE } \mathbb{Z}_p,\ p^n \mathbb{Z}_p \text{ compact \& open}}$

$\mathbb{Q}_p / \mathbb{Z}_p$ = p-torsion in $\mathbb{Q}/\mathbb{Z}$
$GL_1(\mathbb{Q}_p) = \mathbb{Q}_p^\times = \langle p \rangle \times \mathbb{Z}_p^*$
$\mathbb{Z}_p^*$: leading term must be $\neq 0$

Characters of $\mathbb{Q}_p^*$: smooth if identity on some $1 + p^i \mathbb{Z}_p$,
unramified if 1 on $\mathbb{Z}_p^*$
Unramified characters $\leftrightarrow$ chars of $\mathbb{Q}_p^* / \mathbb{Z}_p^* \cong \mathbb{Z}$
$\longleftrightarrow \mathbb{C}^*$.

Note ~~in continuous Eurlidean function on $\mathbb{Q}_p$~~

<u>Def</u> A <u>smooth</u> function on $\mathbb{Q}_p$ is a <u>locally constant</u> function : for each x $f(x) = f(x + p^N a)$ $\forall a$ $N \gg 0$
constant on small opens. (On $\mathbb{Z}_p$: can choose N uniformly in x).
$\iff$ $f : \mathbb{Q}_p \to \mathbb{C}$ continuous wrt <u>discrete</u> topology on $\mathbb{C}$.

$$G = GL_2 \overset{X}{\mathbb{Q}_p} \supset GL_2 \overset{O}{\mathbb{Z}_p} = K$$

invertible 2×2 matrices $\subset Mat_{2\times 2}\, X, O$

$\Rightarrow$ induced topology, open.

$Id + p\, Mat_{n\times n} \mathbb{Z}_p \subset GL_n \mathbb{Q}_p$ compact

$Id + p^n\, Mat_{n\times n} \mathbb{Z}_p$ compact open subgroups, basis of topology $\Rightarrow$ G locally compact topological group

$\rightarrow \exists!$ Haar measure / scalar

$K \subset G$ maximal compact & open

Smooth representations: $\pi : G \longrightarrow GL(V)$ $\underline{\mathbb{C}}$-vector space continuous w.r.t. discrete topology $\Leftrightarrow$

$\forall v \in V$ $K_n = \{ g \in G : g \equiv 1 \bmod p^n \}$ stabilizes v for $n >> 0$.

Unramified reps: $K = GL_2 \mathbb{Z}_p$ ~~acts trivially~~ has fixed vector $V^K \neq 0$

Example: Unramified Principal series representations of $SL_2 \mathbb{Q}_p$.

$$T = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, a \in \mathbb{Q}_p^* \right\} \subset G$$

Unramified ~~Smooth~~ Character $\chi : T \longrightarrow \mathbb{C}^* \Longleftrightarrow s \in \mathbb{C}^*$:

$$\begin{pmatrix} p^i a_r & \\ & p^{-i} a_r^{-1} \end{pmatrix} \longmapsto p^{si} \qquad a_r \in \mathbb{Z}_p^*$$

$$\chi_s : B = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \twoheadrightarrow T \longrightarrow \mathbb{C}^*$$

$$V_s = Ind_B^G \chi_s = \{ f : G \to \mathbb{C} \text{ smooth} : f(n t g) = \chi_k'(t) f(g) \}$$

$\chi_k'(t) = |a|^{\bullet} \chi_k(t)$ analog of shift $s \to s-1$ before

- modular character $\det ad_{\mathfrak{b}} B / \det_{\mathfrak{b}} B$.

The tree of SL_2 over the local field $\mathbb{Q}_p$

Analog of $\mathbb{H} = SL_2\mathbb{R}/SO_2$.

Def $L \subset \mathbb{Q}_p^2 =: V$ is a *lattice* if fin gen $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p^2$ generating V over $\mathbb{Q}_p$ $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = V$

$L \simeq \mathbb{Z}_p^2$ as $\mathbb{Z}_p$-module

eg $\mathbb{Z}_p^2 = \langle e_1, e_2 \rangle$ $\langle e_1, pe_2 \rangle$, $\langle pe_1, pe_2 \rangle$ etc

Consider lattices up to homotheties: rescaling by $\mathbb{Q}_p^*$,

so $\langle e_1, e_2 \rangle \sim \langle pe_1, pe_2 \rangle$

Neighbors: $L_1 \subset L_2$ & $L_2/L_1 \simeq \mathbb{Z}/p$

eg $\langle e_1, e_2 \rangle \supsetneq \langle e_1, pe_2 \rangle \supsetneq \langle pe_1, pe_2 \rangle$

relation becomes symmetric up to homothety, $L_1 \subset L_2 \subset p^{-1}L_1$

$\to$ make graph:

vertices = L up to homothety

edges = neighbor relation.

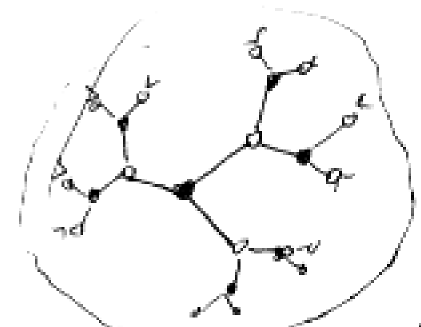
Neighbors of a fixed L = lines in $L/pL \simeq (\mathbb{Z}/p)^2$

$\rightsquigarrow \mathbb{P}^1(\mathbb{F}_p)$ $p+1$ neighbors.

Distance: $d(L_1, L_2) = \text{length}(L_1/L_2)$ after arranging $L_2 \subset L_1$ by homothety. (& $L_2 \not\subset pL_1$)

$L_1/L_2 \simeq \mathbb{Z}/p^n\mathbb{Z}$ n = distance.

Prop This graph is a tree.



color by even/odd: two equiv classes.

Fix L_0 reference (eg $\mathbb{Z}_p^2$)

Each vertex $\longleftrightarrow$ $L \subset L_0$ $L_0/L \simeq \mathbb{Z}/p^n\mathbb{Z}$

$\Longleftrightarrow$ distance n $\longleftrightarrow$ $\mathbb{P}^1(\mathbb{Z}/p^n\mathbb{Z})$ $L_0/p^nL_0 \simeq (\mathbb{Z}/p^n\mathbb{Z})^2$

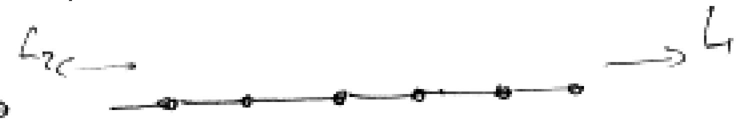
<u>Ends</u> of X : infinite paths from L_0 without backtracking $\Longleftrightarrow$ all paths / equivalence

$\Longleftrightarrow \quad \varprojlim \mathbb{P}(L_0/p^n L_0) = \mathbb{P}^1(\mathbb{Z}_p)$

$= \mathbb{P}(\hat{L}_0) \longleftrightarrow \mathbb{P}^1(V)$: can clear denominators in any line.

$D \in \mathbb{P}^1(V) \Rightarrow$ end $L_n = L_0 p^n + (L_0 \cap D)$

D = unique line contained in $\bigcap L_n$.

<u>Straight path</u>: pair of distinct ends $\longleftrightarrow$ 

$\longleftrightarrow$ decompose $V \cong \ell_1 \oplus \ell_2$ sum of two lines

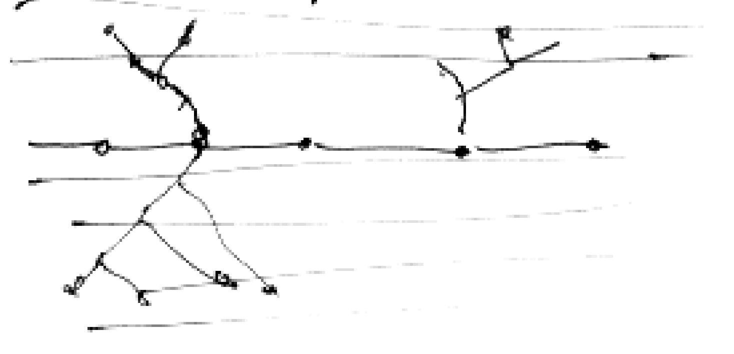
<u>Symmetries</u>: $GL_2\mathbb{Q}_p$ acts on X. Stab $L_0 = GL_2\mathbb{Z}_p$

$\rightsquigarrow$ Vert $X \cong GL_2\mathbb{Q}_p / GL_2\mathbb{Z}_p$

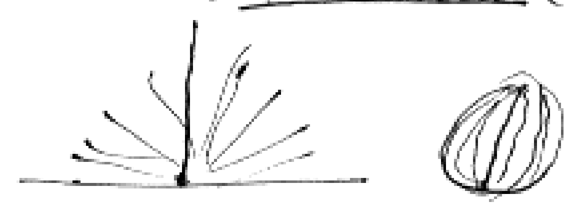
$K = GL_2\mathbb{Z}_p$ preserves distance from origin, orbits circles of given radius $\sim \mathbb{P}^1(\mathbb{Z}/p^n\mathbb{Z})$

$\longleftrightarrow SO_2$ orbits on $\mathbb{H}$.

$T(\mathbb{Q}_p)$: preserves decomposition into lines $\rightsquigarrow$ a straight line in X.



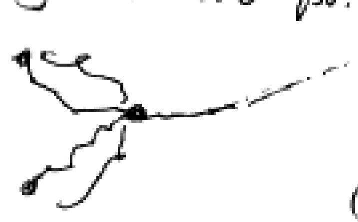
preserves function: <u>distance to line</u>

$\longleftrightarrow \mathbb{H}$: 

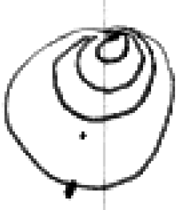
$N(\mathbb{Q}_p)$: stabilizes unique line $\ell \subset V \longleftrightarrow$ end of X

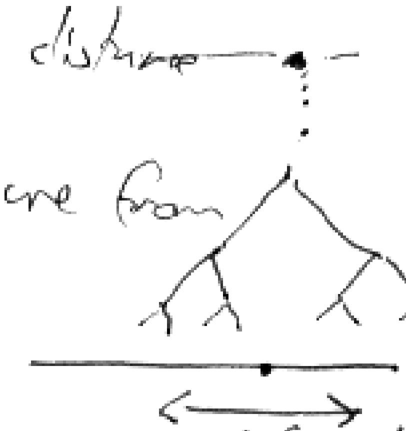
$\longleftrightarrow$ point at ∞ $\quad \mathbb{P}^1(\mathbb{Q}_p) \cong GL_2\mathbb{Q}_p / B_{\mathbb{Q}_p}$

Ex Distance to ∞ : given two points have <u>relative</u> distance to ℓ :



how far both are from common end





N orbits

$\Rightarrow \mathbb{Z}$-torsor of N orbits.

T normalizes $N \Rightarrow$ acts on N orbits

[SL_2 vs GL_2 (or PGL_2) :
$SL_2(\mathbb{Q}_p)$ preserves even/odd coloring of vertices]

Iwahori subgroup: Stabilizer of edge $L_1 < L_2$ $[L_2/L_1 \cong \mathbb{Z}/p]$
is intersection of $\mathrm{Stab}\, L_1 \cap \mathrm{Stab}\, L_2$
$= g \in \mathrm{Stab}\, L_2 :$ $g \bmod pL_2$ preserves L_1

$L_0 = \mathbb{Z}_p^2 :$ $g \in GL_2 \mathbb{Z}_p !$ $g \bmod p$ upper-triangular

$\Longleftrightarrow$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2 \mathbb{Z}_p$: $c \equiv 0\ (p)$.

$\longleftrightarrow$ analog of $\Gamma_0(p)$.

$SL_2 \mathbb{Q}_p = SL_2 \mathbb{Z}_p \underset{I}{*} SL_2 \mathbb{Z}_p$  Bass-Serre theory

<u>Hecke operators, tree version</u>: f compactly supported fn on tree

$\Theta_p f(x)\ T_p f(x) = \sum_{d(y,x)=1} f(y)$ average over nearest nbrs

$\Theta_{p^n} f(x) = \sum_{d(y,x)=n} f(y)$ $\quad T_{p^n} f(x) = \Theta_{p^n} f(x) + T_{p^{n-2}} f(x)$

$\boxed{T_p T_{p^n} = T_{p^{n+1}} + p T_{p^{n-1}}}$ $\Longrightarrow$

T_{p^n}'s form a commutative algebra.

$\sum_0^\infty T_{p^n} \lambda^n = \frac{1}{1 - T_p \lambda + p \lambda^2}$

Analog of Laplacian: f harmonic $\Delta f = 0$ $\Longleftrightarrow$ $T_p f = (p+1) f$
average of its nearby values.

$\Theta_{p^n} f(L_0)$: average of values on $GL_2 \mathbb{Z}_p$-orbit...
Operators labelled by $GL_2\mathbb{Z}_p \backslash X$ $\longleftrightarrow$ $SO_2 \backslash \mathbb{H} = \mathbb{R}_+$ distance

$\qquad\qquad\qquad\qquad\qquad\qquad ||\qquad\qquad\qquad\qquad ||$

On $\mathbb{H}$: $\delta_t * f(x) = \int_{|y-x|=t} f(y)\, dy$ $\qquad GL_2\mathbb{Z}_p \backslash GL_2\mathbb{Q}_p / GL_2\mathbb{Z}_p$ $\qquad SO_2 \backslash SL_2\mathbb{R} / SO_2$

or $(h(t) * f)(x) = \int_{\mathbb{R}} h(t) \cdot f(x)\, dt$

Convolution & Hecke operators

$$\begin{tikzcd} & G \times G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[d, "m"] & \\ G & G & G \end{tikzcd}$$

$$\mathbb{C}G: \quad f * h(x) = m_*(f \boxtimes h)(x)$$
$$= \int_{m^{-1}(x)} f\,h\,dg = \int_G f(g)\,h(g^{-1}x)\,dg$$

$$\begin{tikzcd} & K\backslash G \times_K G/K \arrow[dl] \arrow[dr] \arrow[d] & \\ K\backslash G/K & K\backslash G/K & K\backslash G/K \end{tikzcd}$$

$\mathbb{C}[K\backslash G/K]$ subalgebra under convolution

$$\begin{tikzcd} & \Gamma\backslash G \times_K G/K \arrow[dl] \arrow[dr] \arrow[d] & \\ \Gamma\backslash G/K & \Gamma\backslash G/K & K\backslash G/K \end{tikzcd}$$

action on $\mathbb{C}[\Gamma\backslash G/K]$:

the principal K bundle $\begin{matrix}\Gamma\backslash G \\ \downarrow \\ \Gamma\backslash G/K\end{matrix}$,

take associated G/K bundle $\Gamma\backslash G \times_K G/K$.

K-invariant function on G/K (e.g. char. function of a coset) gives function on $\Gamma\backslash G \times_K G/K$, can integrate against it.

Hecke operators via correspondences: $(g_1, g_2K) \mapsto (g_1 k, k^{-1} g_2 K)$

$$\begin{array}{ccccc} & \xleftarrow{\pi} & G \times_K G/K & \xrightarrow{m} & \\ G/K \times K\backslash G/K & & & & G/K \\ f \quad\quad h & & & & \end{array}$$

$\{g_1K, g_2K, g : gg_1K = g_2K\}$

G/K

$g \in G/K$

Given $h \in K\backslash G/K$ get subset of G/K, all things in relative position h to g.

Commutes with left G action:

$h : \Gamma\backslash G/K \dashrightarrow \Gamma\backslash G/K$ any Γ.

Given $h \in K\backslash G/K$ or more generally h function on $K\backslash G/K$ $\Rightarrow$ operator $\mathrm{Fun}(G/K) \circlearrowleft$

$h * f\,(x) = m_*\,(\pi^*(f \otimes h))$

$$\begin{array}{ccccc} & \overset{\pi}{\swarrow} & G \times_K G/K & \overset{m}{\searrow} & \\ G/K & & \downarrow \pi_2 & & G/K \\ & & K\backslash G/K & & \end{array}$$

associated G/K bundle over G/K

$\Rightarrow$ any K-invariant function on G/K gives function on $G \times_K G/K$

What is all this? part of convolution of functions on G:

$K = 1$: $G \xleftarrow{\pi_1} G \times G \xrightarrow{m} G \iff G \xleftarrow{\pi_1} G \times G \xrightarrow{\pi_2} G$, $G \times G \xrightarrow{m} G$

$\mathbb{C}G$: $f * h(x) = \int_G f(g)\, h(g^{-1}x)\, dg$

$= \int_{m^{-1}(x)} f\, h\; dg \quad = m_*(f \boxtimes g)$

$\mathbb{C}G \supset (\mathbb{C}G)^K = \mathbb{C}[G/K]$ not algebra

$\mathbb{C}G \supset {}^K\mathbb{C}G^K = \mathbb{C}[K\backslash G/K]$ subalgebra:

$K\backslash G/K \leftarrow K\backslash G \times_K G/K \rightarrow K\backslash G/K$

Hecke algebras

$K \subset G$ $\quad$ V G-rep $\Rightarrow$ $V^K \subset V$ K-invariants

Ask: what preserves V^K?

answer 1: $N(K) \subset G$ normalizer

$V \longrightarrow V^K$ $\quad$ e.g. highest weight theory!

G-rep $\longrightarrow N(K)$-rep $\quad$ $K = N \subset SL_2$

V^N = highest weight vectors. $\quad$ $\mathrm{Norm}(N)/N = T$

$T \circlearrowright V^N$ gives weight

irred SL_2-rep $\longleftrightarrow$ irred T-rep $\Leftrightarrow$ weight

Often K has no normalizer, eg $SL_2\mathbb{Z}_p \subset SL_2\mathbb{Q}_p$.

Ask again: look in $\mathbb{C}G$ not G.

$v \in V^K$ $\quad$ $f \in \mathbb{C}G$ need $f \cdot v \in V^K$

$(k * f) \cdot v = f \cdot v = (f * k) \cdot v$

$\Rightarrow$ want $f \in \mathbb{C}G^{K \times K} =: \mathcal{H}(G,K)$.

More abstractly: $V^K = \mathrm{Hom}_K(1, V) = \mathrm{Hom}_G(\mathrm{Ind}_K^G 1, V)$

$= \mathrm{Hom}_G(\mathbb{C}[G/K], V)$ $\quad$: $V^K \neq 0 \Leftrightarrow V$ appears in $\mathbb{C}[G/K]$!

$1 \mapsto v$

What acts on this universally?

$\mathrm{Hom}_G(\mathbb{C}[G/K], \mathbb{C}[G/K]) = \mathbb{C}[G/K]^K = \mathbb{C}[K\backslash G/K]$

Idea: $\left\{\begin{array}{c}\text{irreducible reps of } G \text{ with} \\ V^K \neq 0\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{irreducible reps} \\ \text{of } \mathcal{H}(G,K)\end{array}\right\}$

$V \longmapsto V^K$

If K ~~large~~ small, tend to have K-invariants

If K large, $\mathcal{H}(G,K)$ simple !

$\left\{\begin{array}{c}G\text{-reps generated by} \\ V^K\end{array}\right\} \xrightarrow{\sim} \left\{\mathcal{H}(G,K)\text{-reps}\right\}$

<u>p-adic groups</u> V smooth rep $\Rightarrow$ $V = \bigcup_n V^{G(\mathbb{Z}_p)_n}$

$G(\mathbb{Z}_p)_n = \{ g \equiv 1 \bmod p^n \}$

V smooth admissible: $V^{G(\mathbb{Z}_p)_n}$ <u>finite</u> <u>dimensional</u>

<u>Group algebra</u>: need only <u>smooth</u> (locally constant) functions $\mathbb{C}G(\mathbb{Q}_p)$

$\mathcal{H}(G(\mathbb{Q}_p), G(\mathbb{Z}_p)_n)$ bi-invariant smooth functions

$\mathrm{Rep}\, G(\mathbb{Q}_p) = \bigcup_n$ reps with $G(\mathbb{Z}_p)_n$-fixed vectors
$\longleftrightarrow \mathcal{H}_n$-reps.

<u>Unramified</u> reps: $V^{G(\mathbb{Z}_p)} \neq 0$

$\mathcal{H}_{sph}$ spherical Hecke algebra $= \mathbb{C}G(\mathbb{Q}_p)^{G(\mathbb{Z}_p)\times G(\mathbb{Z}_p)}$

$= \mathbb{C}[\text{tree}]^{G(\mathbb{Z}_p)} = \mathrm{End}_{G(\mathbb{Q}_p)} \mathbb{C}[\text{tree}]$

<u>Commutative</u> algebra. $\simeq \mathbb{C}[T_p]$

$\left\{ \begin{matrix} \text{irreducible unramified} \\ G\text{-reps} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{irreducible } \mathcal{H}_{sph}\text{-} \\ \text{reps} \end{matrix} \right\}$

$\updownarrow$

eigenvalues of $T_p : V^{G(\mathbb{Z}_p)} \simeq \mathbb{C}$

How to describe this? want coset representatives for $K \backslash G(\mathbb{Q}_p) / K$: take $\begin{pmatrix} p^n & \\ & p^m \end{pmatrix} \in GL_2\, \mathbb{Q}_p$

up to [illegible] $n \sim m$ so $\begin{pmatrix} p^n & \\ & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & \\ & p^n \end{pmatrix}$

$\Big($ $SL_2 \mathbb{Z}_p$: want $\begin{pmatrix} p & \\ & p^{-1} \end{pmatrix} \sim \begin{pmatrix} p^2 & \\ & 1 \end{pmatrix}$ not $\begin{pmatrix} p & \\ & 1 \end{pmatrix}$ $\Big)$ $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ Weyl group

= coweights $/W$.

<u>Satake isomorphism</u> $s \in \mathbb{C}^* = \mathrm{Hom}^{unram}(\mathbb{Q}_p^* \twoheadrightarrow \langle p^n \rangle, \mathbb{C}^*)$

$\Rightarrow V_s = \mathrm{Ind}_B^{GL_2 \mathbb{Q}_p} \mathbb{C}_{s,1}$.
spherical principal series.

$V_s^{G(\mathbb{Z}_p)} \cong \mathbb{C}$! spherical vector

$$\Rightarrow \quad \mathcal{H}_{sph} \longrightarrow \mathbb{C}[s, s^{-1}] = \mathbb{C}[T_{\mathbb{C}}^\vee]$$

- real analog: Harish-Chandra homomorphism
  Look at action of center $Z < U\mathfrak{g}$ on principal series reps V_s ($\Leftrightarrow$ on their spherical vectors) or Verma modules V_λ ($\Leftrightarrow$ on their h.w. vector)

$$\begin{tikzcd} Z \arrow[r] \arrow[dr, "\sim"'] & \mathbb{C}[\mathfrak{h}^\vee] \\ & \mathbb{C}[\mathfrak{h}^\vee]^W \arrow[u, hook] \end{tikzcd} \qquad \mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$$

intertwiners $\Rightarrow$ W-invariance.

- p-adic setting:

$$\begin{tikzcd} T(\mathbb{Q}_p) \arrow[r] & T(\mathbb{Q}_p)/T(\mathbb{Z}_p) \arrow[r] & \mathbb{C}^* \\ \Lambda \arrow[u, hook] \arrow[ur, "\sim"'] & & \end{tikzcd}$$

lattice of coweights $\begin{pmatrix} p^{i_1} & & \\ & p^{i_2} & \\ & & \ddots \; p^{i_n} \end{pmatrix}$

$\mathrm{Hom}(\Lambda, \mathbb{C}^*) = T_{\mathbb{C}}^\vee$ dual torus

$$\begin{tikzcd} \mathcal{H}_{sph} \arrow[r] \arrow[dr, "\sim"', "\text{intertwiner}"] & \mathbb{C}[T_{\mathbb{C}}^\vee] \\ & \mathbb{C}[T_{\mathbb{C}}^\vee]^W = \mathrm{Rep}\, G_{\mathbb{C}}^\vee \arrow[u] \end{tikzcd}$$

Cosets $\Lambda/W \leftrightarrow K \backslash G / K$

Character of $\mathcal{H}_{sph} \Leftrightarrow$ semisimple conj class in $G^\vee$:
$\begin{pmatrix} s_1 & \\ & s_2 \end{pmatrix} \longleftrightarrow V_{s_1, s_2}$. $\mathcal{H} = \mathbb{C}[G^{\vee, ss}]^{G^\vee} = \mathrm{Rep}\, G_{\mathbb{C}}^\vee$

Adèles $\mathbb{A} = \prod' \mathbb{Q}_p \times \mathbb{R}$

Topology: $\prod \mathbb{Z}_p \times \mathbb{R}$ open with product topology
Convergence: pointwise convergence + require eventually $\forall p$ $(x)_p \in (x)_p + \mathbb{Z}_p$.

$\mathbb{A}$ locally compact.
Not so bad: let $\mathcal{O}_{\mathbb{A}} = \prod \mathbb{Z}_p < \mathbb{A}$

$\mathbb{Z} \to \mathcal{O}_{\mathbb{A}}$ $\quad n \mapsto (n, n, \dots)$
Dense: ~~approximate by integers~~ $\mathcal{O}_{\mathbb{A}} \simeq \hat{\mathbb{Z}} = \varprojlim$ finite quotients of $\mathbb{Z}$ $= \prod \varprojlim$ finite p-quotients $= \prod \mathbb{Z}_p$.

$\mathbb{Q} \to \prod' \mathbb{Q}_p$ similarly dense
(completion wrt topology: basis of nbhds of 0 = all subgroups of $\mathbb{Q}$).

OTOH: $\mathbb{Q} \to \mathbb{A}$ $\quad r \mapsto (r, r, \dots r \dots)$
is discrete: if $(r_n, r_n, \dots) \to 0$
$\Rightarrow$ eventually all $\in \mathbb{Z}_p$ $\Rightarrow r_n \in \mathbb{Z}$
$\Rightarrow (r_n)_\infty \not\to 0$

$\mathbb{A}/\mathbb{Q}$ compact: fundamental domain $F = \{0 \le a_\infty < 1, \forall |a_p|_p < 1\}$

$\mathbb{A}/\mathbb{Q} = \mathbb{Q}^\vee$ pontryagin dual $\quad$ pairing $\langle a,b \rangle = \exp\left(2\pi i \cdot \sum a_p b_p - a_\infty b_\infty\right)$
$(\mathbb{Q}/\mathbb{Z})^\vee = \mathcal{O}_{\mathbb{A}} = \hat{\mathbb{Z}}$

... note exp defined on $\sum a_p b_p$ up to integer $\to$ finite sum of rationals

Multiplicative: $\mathbb{A}^\times / \mathcal{O}^\times = \prod' \mathbb{Z} \times \mathbb{R}_+$

[story over $\mathbb{F}_q$]

Intro to Langlands

Describe decomposition of $L^2(SL_2\mathbb{Z}\backslash SL_2\mathbb{R}) \circlearrowleft SL_2\mathbb{R}$
or $L^2(\Gamma_0(N)\backslash SL_2\mathbb{R})$

... or really corresponding automorphic form space $\longleftrightarrow$ $(\mathfrak{g}, K)$ module:

$\Gamma \quad \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$
$\Gamma_0 \quad \begin{pmatrix} \cdot & \cdot \\ & \cdot \end{pmatrix} \quad \text{mod } N$
$\Gamma_1 \quad \begin{pmatrix} 1 & \cdot \\ & 1 \end{pmatrix}$

- C^∞ · K-finite · Z-finite · moderate growth

Discrete series part: k-forms on $\Gamma\backslash SL_2\mathbb{R}/SO_2 = K_\infty$

$$\Gamma(n)\backslash SL_2\mathbb{R}/K_\infty = SL_2\mathbb{Q}\backslash SL_2\mathbb{A}/K(n) \times K_\infty$$

$K(N) \subset \prod SL_2(\mathbb{Z}_p)$ satisfy congruence mod n.

So we're looking at $K(N) \times K_\infty$-invariant functions on $SL_2\mathbb{Q}\backslash SL_2\mathbb{A}$ finite volume homogeneous space.

Automorphic forms: C^∞, K_∞ finite, loc constant, $K(N)$ finite ~~space~~, moderate growth : $\mathcal{A}(SL_2\mathbb{Q}\backslash SL_2\mathbb{A})$

— analog of $\mathfrak{g}, K$ module over $\mathbb{A}$.

$$\mathcal{A}_{SL_2} = \bigcup_N \{\mathcal{A}_{SL_2, n} = K(N)\text{-invariant functions}\} \quad \bigotimes_{p\nmid N} \mathcal{H}_p \circlearrowleft$$

$$= \mathcal{A}(\Gamma(N)\backslash SL_2\mathbb{R}) = \mathcal{A}(SL_2\mathbb{Q}\backslash SL_2\mathbb{A}/\prod_{p\nmid N} SL_2\mathbb{Z}_p \times K_{(N)})$$

Irreducible automorphic rep

$V = \otimes' V_p \qquad V_p$ unramified $p \notin \{\text{div } N\}$

i.e. $V_p^{SL_2\mathbb{Z}_p} \simeq \mathbb{C} v_p$. $\quad V = \{\bigotimes_p w_p : w_p = v_p \text{ all but fin many } p\}$

Studying $SL_2\mathbb{A}$ action on $\mathcal{A}_{SL_2}(N)$
$\rightsquigarrow$ studying $\bigotimes_p \mathcal{H}_{SL_2\mathbb{Q}_p, SL_2(\mathbb{Z}_p, N(p))}$ all but fin many commutative

Satake isomorphism Classify unramified $SL_2\mathbb{Q}_p$ reps

$$V_{s^2} = \mathrm{Ind}_B^{SL_2\mathbb{Q}_p}\left\{ \begin{pmatrix} p & \\ & p^{-1}\end{pmatrix} \mapsto s^2 \in \mathbb{C}^* \right\}$$

$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$: $V_{s^2} \leftrightarrow V_{s^{-2}}$ V_s parametrized by $\mathbb{C}^*/\mathbb{Z}_2$

$\Leftrightarrow$ $\begin{pmatrix} s^2 & \\ & s^{-2}\end{pmatrix} \in PSL_2\mathbb{C}$ up to conjugacy:
semisimple conjugacy class in $PSL_2\mathbb{C}$ = Langlands dual group

GL_2: $\begin{pmatrix} p & \\ & 1\end{pmatrix} \mapsto s_1$ $\begin{pmatrix} 1 & \\ & p\end{pmatrix} \mapsto s_2$

$\Longleftrightarrow \left\{\begin{pmatrix} s_1 & \\ & s_2\end{pmatrix}\right\} \in GL_2\mathbb{C}^{ss}/\mathrm{conj}$

Recall $\displaystyle\sum T(n) n^{-s} = \prod_p \frac{1}{1 - T(p)p^{-s} + p^{-2s}}$

$$= \prod_p \frac{1}{\det\left(1 - p^{-s}\begin{pmatrix} \eta(p) & \\ & \eta(p)^{-1}\end{pmatrix}\right)}$$

V irred automorphic rep level N, $D(k)$ at infinity
$\leftrightarrow$ weight k level N holomorphic cusp form

$\leftrightarrow$ $\{a_p\}_{p \notin S = \mathrm{div}\, N}$ $\Leftrightarrow$ $\{\lambda_p\}$ $T(p)$ eigenvalues.
+ info at ramified places

$\longrightarrow$ $L(V,s) = L(f,s) = \prod \dfrac{1}{\det\left(1 - p^{-s}\begin{pmatrix} a_p & \\ & a_{-p}\end{pmatrix}\right)}$
$= \sum \frac{a_n}{n^s}$

functional equation, analytic continuation.

Which $\{a_p\}$ appear?

So should look for <u>Hecke eigenfunctions</u>

f modular form s.t. $T_p f = a_p f$

Miracle: Fourier expansion $f = \sum a_n e^{inx}$ $\qquad \prod_p \frac{1}{\det\left(1-p^{-s}\begin{pmatrix} a_p & -1 \\ & 0 \end{pmatrix}\right)}$

$a_n = T_n$-eigenvalue!

$\Rightarrow$ L-series $L(f,s) = \sum \frac{a_n}{n^s} = \prod_p \frac{1}{\left(1-a_p p^{-s-\frac{k-1}{2}} + p^{-2s}\right)}$

Recall $\sum T_{p^n} \lambda^n = \frac{1}{1-T_p\lambda + p\lambda^2}$ $\qquad$ Euler product

<u>Langlands</u> Which irred reps occur in $L^2(SL_2\mathbb{Q} \backslash SL_2\mathbb{A})$?

~~$V = \otimes \otimes' V_p$~~

(more precisely in space of <u>automorphic fns</u>:
($\mathfrak{sl}_2\mathbb{R}$, SO_2)-module at ∞
K-finite $K \subset G(\mathbb{A}_f)$ compact open,
$Z(U\mathfrak{g})$-finite, moderate growth)

Look for reps unramified outside given finite set S of places $\Rightarrow$ have $SL_2\mathbb{Z}_p$-fixed vectors $p \notin S$.

$V = \otimes' V_p$ $\qquad f = \otimes f_p$, all but fin many f_p = the spherical vector.

$\Rightarrow a_p$ T_p-eigenvalues all $p \notin S$.

Which $\{a_p\}$ can occur??

A: Galois reps:

$\rho : \mathrm{Gal}\,\bar{\mathbb{Q}}/\mathbb{Q} \longrightarrow PGL_2$

Strong approximation:

$$\mathbb{Q}\cdot\mathbb{R} \subset \mathbb{A} \text{ dense}$$

$\Rightarrow$ take $O < \mathbb{A}$ any open subgroup, eg $O_{\mathbb{A}}$ $\Rightarrow$
$\mathbb{A} = O\,\mathbb{Q}\,\mathbb{R}$ or $\mathbb{R} \twoheadrightarrow \mathbb{A}/O\,\mathbb{Q}$

F number field $\Rightarrow$ $\mathbb{A}^\times/\mathbb{Q}^\times O_{\mathbb{A}}^\times$ = ideal class group
$\#|\ \ | =$ class number of F, finite.

Theorem: $SL_2\mathbb{R}\, SL_2\mathbb{Q} \subset SL_2\mathbb{A}$ dense

Corollary: $SL_2\mathbb{R} \twoheadrightarrow {}_{SL_2\mathbb{Q}}\backslash SL_2\mathbb{A}/SL_2 O_{\mathbb{A}}$ or $K = \prod' SL_2(\mathbb{Z}_p)_n$

$$SL_2\mathbb{Q}\backslash SL_2\mathbb{A}/SL_2 O_{\mathbb{A}} = SL_2\mathbb{Q}\cap K\backslash SL_2\mathbb{R} = \Gamma(N)\backslash SL_2\mathbb{R}$$

$$SL_2\mathbb{Q}\backslash SL_2\mathbb{A}/K\cdot K_\infty = SL_2\mathbb{Z}\backslash SL_2\mathbb{R}/SO_2 = \Gamma\backslash\mathbb{H}.$$

In general $G(\mathbb{Q})\backslash G(\mathbb{A})/K = \coprod_{\text{finite}} \Gamma_i\backslash G(\mathbb{R})$

$$\Rightarrow L^2(\Gamma_N\backslash SL_2\mathbb{R}) = L^2(SL_2\mathbb{Q}\backslash SL_2\mathbb{A}/K_N)$$

Corollaries

$V = L^2(SL_2\mathbb{Q}\backslash SL_2\mathbb{A})$ unitary rep of $SL_2\mathbb{A}$.
$\cup$
$V^{\prod K_{p,n}} \circlearrowleft \bigotimes \mathcal{H}_{p,n}$ Hecke algebras

e.g. $L^2(\Gamma\backslash\mathbb{H}) \circlearrowleft \bigotimes\langle T_p\rangle$ commutative algebra gen by T_m all p, commuting with $SL_2\mathbb{R}$.

$$\begin{array}{ccc} & \text{Hecke}_N & \\ \swarrow & & \searrow \\ \Gamma\backslash\mathbb{H} & & \Gamma\backslash\mathbb{H}\end{array} \qquad = \{E, G\subset E(\mathbb{Q})\ \ G\cong\mathbb{Z}/N\} \qquad \begin{array}{ccc} \swarrow & & \searrow \\ E & & E/G\end{array}$$

Hecke operators & Euler products

$$\sum_{n\geq 1} T(n) n^{-s} = \prod_p (1 - T(p)p^{-s} + p^{(k-1)-2s})^{-1}$$

f normalized cusp form, Hecke eigenform $T(n)\cdot f = \lambda(n) f$

$$\Rightarrow L(f,s) = \sum a_n n^{-s} = \sum \lambda_n n^{-s} = \prod (1 - \lambda_p p^{-s} + p^{(k-1)-2s})^{-1}$$

$\Rightarrow$ ~~f~~ $f = \sum a_n q^n$ q-expansion.

$$\lambda_p \cdot a_n = \begin{cases} a_{pn} & p \nmid n \\ a_{pn} + p^{k-1} a_{n/p} & p \mid n \end{cases}$$

i.e $\lambda_p a_1 = a_p$.
Normalize : $a_1 = 1 \Rightarrow a_p = \lambda_p$.