Marco Aldi - KP & Witten's Conjecture

Note Title 5/15/2007

$\overline{\mathcal{M}}_{g,n}$ = stable curves w/ n marked points

$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$

$\mathcal{L}_i$ = i^{th} cotangent line bundle

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \qquad \sum d_i = 3g-3+n$$

$$F(t_0, t_1, \ldots) = \sum \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \frac{t_{d_1} \cdots t_{d_n}}{|\mathrm{Aut}(d_1, \ldots, d_n)|}$$

<u>Witten Conjecture / Kontsevich's Theorem</u>

e^F τ-function for KdV

- i.e. $u = \frac{\partial^2 F}{\partial t_0^2}$ satisfies KdV:

$$\frac{\partial u}{\partial t_n} = R_{n+1}\left(u, \frac{\partial u}{\partial t_0}, \frac{\partial^2 u}{\partial t_0}, \ldots\right)$$ & R's determined recursively

String equation: $\frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i}$

$$\Rightarrow \langle \tau_0 \prod_1^n \tau_{d_i} \rangle = \sum_{j=1}^{n} \langle \prod_1^n \tau_{d_i - \delta_{ij}} \rangle$$

The theorem $\Rightarrow$ can reconstruct F recursively using KdV & string equation:
$u(t_0, t_1, \ldots)$ determined by $u(t_0, 0, 0, \ldots)$
& string equation determines $u(t_0, 0 \ldots)$

Motivation: string theory on point should be 2d topological gravity, as should theory of random surfaces
- encode by 't Hooft matrix integral

$$Z = \int dM\, e^{-\mathrm{Tr}\, V(M)} \qquad V \text{ polynomial}$$

To solve diagonalize: $M = U \Lambda U^{-1}$, U unitary
$\Lambda = \mathrm{diag}(s_1, \ldots s_n)$

$$dM = \text{const} \cdot dU \; ds_1 \ldots ds_n \prod_{i<j} (s_i - s_j)^2$$

Basis of orthogonal polynomials (monic)

$$\int ds\, e^{-V(s)}\, P_r(s)\, P_k(s) = h_r\, \delta_{k,r}$$

$$\Rightarrow \quad Z = N!\, h_1 \cdots h_{N-1} \qquad \hat{P}_r = \frac{P_r}{\sqrt{h_r}}$$

Write $V(s) = \sum w_i s^i$

$\mathcal{S} : \mathbb{C}[s] \circlearrowleft$

defined by $\mathcal{S}\hat{P}_r(s) = s\hat{P}_r(s) = \sum S_{r,k}\hat{P}_k(s)$

$$= \frac{\sqrt{h_{r+1}}}{\sqrt{h_r}}\hat{P}_{r+1} + \text{const}\,\hat{P}_r + \frac{\sqrt{h_r}}{\sqrt{h_{r-1}}}\hat{P}_{r-1}$$

$S_{r,k} = S_{k,r}$ & S is tridiagonal:

$$S = \|S_{r,k}\| = \begin{pmatrix} \ddots & \ddots & & 0 \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}$$

(by integration by parts)

Vary w_i: $\dfrac{\partial \hat{P}_k}{\partial w_i} = \sum_l \left(\mathcal{O}_{(i)}\right)_k \hat{P}_r$

$$\frac{\partial S}{\partial w_i} = [\mathcal{O}_{(i)}, S]$$

$\leadsto$ family of commuting flows.

$$\lim_{N\to\infty} S = \frac{d^2}{dx^2} + u(x),$$

flows become KdV flows

String equation: $[S, T] = 1$

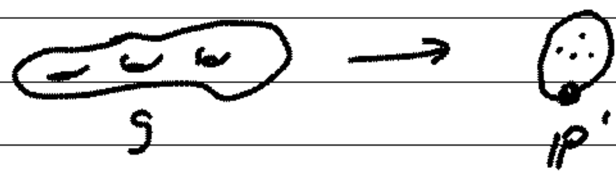
$T = \frac{d}{ds} \curvearrowright \mathbb{C}[s]$.

Okounkov-Pandharipande proof: uses Hurwitz numbers & Eguchi-Xiong matrix model

Mirzakhani proof: volumes of WP moduli

$\boxed{\text{Kazarian-Lando}}$ JAMS ... also uses Hurwitz numbers

Consider



ramification profile: $(b_1, \dots b_n)$ at one point, all other ramifications simple.

n=2 sheets

$m = \#$ ramification points $= 2g - n + \sum b_i$

$$h_{g,b_1,\dots,b_n} = \frac{\#\text{ covers}}{|\mathrm{Aut}|}$$

$$H(\beta, p_1 \dots p_n) = \sum h_{g,b_1,\dots,b_n} \frac{p_{b_1} \cdots p_{b_n}}{n!} \frac{\beta^m}{m!}$$

<u>Theorem</u> e^H is a KP tau-function

<u>Lemma</u> $$\frac{\partial e^H}{\partial \beta} = \left(\frac{1}{2} \sum_{i,j} (i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} + ij\, p_{i+j} \frac{\partial}{\partial p_i \partial p_j} \right) e^H$$

$$=: A e^H$$

... formula describes bringing simple ramification point to the special point:
it can collide in 2 ways:
can either join two ramification points
or cause one to split.

Sato Formalism

Fock space $\mathcal{F} = \bigoplus \mathbb{C} s_\lambda \quad \lambda = (\lambda_1, \lambda_2, \dots)$

$\lambda_1 \geq \lambda_2 \geq \dots \geq 0$

- $\mathcal{F} = \mathbb{C}[[p_1, p_2, \dots]]$

  via identifying s_λ with Schur polynomials

  $$s_\lambda = \det \| s_{\lambda_j - j + i} \|$$

  $$s_0 + s_1 z + s_2 z^2 + \dots = e^{p_1 z + p_2 z^2 + \dots}$$

- $\mathcal{F} = \Lambda^{\infty/2} \mathbb{C}[z, z^{-1}]$

  $$s_\lambda = z^{k_1} \wedge z^{k_2} \wedge \dots \quad \text{where } k_i = i - \lambda_i$$

$\tau = \varphi_1(z) \wedge \varphi_2(z) \wedge \dots$

where φ_i are linearly independent

& $\varphi_i(z) = z^i +$ lower terms

Miracle s_λ are eigenvectors for joining operator A,

$$A s_\lambda = f(\lambda) s_\lambda$$

$$f(\lambda) = \frac{1}{2} \sum_{i}^{\infty} \left[(i - \tfrac{1}{2} - \lambda_i)^2 - (i - \tfrac{1}{2})^2 \right]$$

$$e^H = \sum_\lambda c_\lambda e^{f(\lambda)\beta} s_\lambda$$

At $\beta = 0$ find $c_\lambda = s_\lambda(1, 0, \dots)$

$\rightsquigarrow$ .... get KP τ function H.

<u>ELSV</u>: relates Hurwitz #s to intersection #s on $\mathcal{M}_{g,n}$

$$\overline{\mathcal{M}}(\mathbb{P}^1, d) \xrightarrow{\text{branching}} \mathrm{Sym}^n \mathbb{P}^1$$

$$h = \int_{\overline{\mathcal{M}}(\mathbb{P}^1, d)} br^*(\text{partition}) \dots$$

$\rightsquigarrow$ invert to calculate $\langle \tau_{d_1} \dots \tau_{d_n} \rangle$ in terms of $h_{g, b_1, \dots b_n}$

- translate the KP τ fn H to KdV τ fn F.