M. Ando – Circle-equivariant Sigma-orientations & Reps of loop groups 3/4/03

An Elliptic cohomology theory:
- E even periodic ring spectrum, homotopy commutative
- C elliptic curve over $\pi_0 E$

[Note E is automatically complex orientable]

- Isom of formal groups $t: G_E \xrightarrow{\sim} \hat{C}$ $\quad (G_E \sim E(\mathbb{CP}^\infty))$

Ando-Hopkins-Strickland: construct a map of ring spectra

$\sigma: MU\langle 6\rangle \longrightarrow E \qquad \sigma = \sigma(E, C, t)$ assoc to these data

the Sigma orientation

$MU\langle 6\rangle$ = bordism spectrum of $\mathbb{C}$-mflds with $c_1 = c_2 = 0$

e.g. When $(E, C, t) = K[[q]]$, Take curve $C/\mathbb{Z}[[q]]$, (canonical) + t

Then $\sigma: MU\langle 6\rangle \longrightarrow K[[q]]$ factors through MU, get characteristic series

$$\begin{array}{ccc} MU\langle 6\rangle & \longrightarrow & K[[q]] \\ \downarrow & \nearrow & \\ MU & & \end{array}$$

for an MU-orientation, which in this case is the Weierstrass σ-function $\quad \sigma(u,q) = (1-u)\prod \frac{(1-q^n u)(1-q^n u^{-1})}{(1-q^n)^2}$

Here $C = \mathbb{C}^\times / q^{\mathbb{Z}} \qquad 0 < |q| < 1$,

σ fn on $\mathbb{C}^*$ vanishing to order one on $q^{\mathbb{Z}}$

Associated map to cobordism is the Witten genus:

$$\begin{array}{ccc} \pi_* MU\langle 6\rangle & \longrightarrow & \text{modular forms}/\mathbb{Z} \text{ for } SL_2\mathbb{Z} \\ \downarrow & & \downarrow \\ \pi_* MU & \xrightarrow{\text{Witten}} & \mathbb{Z}[[q]] \end{array}$$

$\Rightarrow$ account for modularity of the Witten genus

Now ask for equivariant analog: is there a map $MU\langle 6\rangle_{\mathbb{T}} \longrightarrow E_{\mathbb{T}}$?

M ~~spin~~ manifold (eg SU6-manifold), T tangent bundle

V spin vector bundle over M.

Define $S_t V = \sum_{k\geq 0} t^k S^k(V^{\mathbb{C}})$

$\Lambda_t V = \sum{}'' \Lambda^k(V^{\mathbb{C}})$

give exponential characteristic classes $KO(X) \longrightarrow K(X)[[t]]$

Witten genus twisted by V $\quad W(M,V) = \mathrm{ind}\big(\not{D} \otimes \bigotimes S_{q^n}(\underbrace{\bar{T}}_{T - \mathrm{rk}\,T \cdot \mathbb{R}})\big) \otimes \cdots$

$\not{D}$ = Dirac operator of tangent bundle

reduced bundle

D contributes term $u^{\frac{1}{2}}-u^{-\frac{1}{2}}$ to char series, $S_{q^n} V^{\mathbb{C}}$ will contribute $(1-q^n u)(1-q^n u^{-1})$
denominator $(1-q^n)^2$ comes from passing from T to reduced bundle $T - \mathrm{rk}\,T \cdot \mathbb{C}$

$$\mathbb{Z}[[q]] \ni W(M,V) = \mathrm{ind}\left(D \otimes \bigotimes S_{q^n}(T-\dim_{\mathbb{C}}) \otimes \Delta_+(V) \otimes \bigotimes_{n\geq 1} \Lambda_{q^n} V\right)$$

– sufficiently geometric that can replace index by equivariant index, get element instead of $R[\mathbb{T}][[q]]$

<u>Rigidity theorem</u> If equivariant c_2 class $c_2(T_{\mathbb{T}} - V_{\mathbb{T}}) = 0$
then $W_{\mathbb{T}}(M,V) = W(M,V)$ constant Laurent polynomials
– indep of rep ring variables

$$c_2 \in H^4(M_{\mathbb{T}} = E\mathbb{T} \times_{\mathbb{T}} M)$$

[Why c_2? ...] Closed string in M, calculate one loop amplitude, couple left moving fermions to T & right movers to V.
c_2 is obstruction to getting CFT – Green-Schwarz mechanism take T & replace by $T-V$..
Pfs: Witten, Bott-Taubes, Liu
More elliptic cohomological: Rosu rigidity for Ochanine genus
Ando-Basterra for Witten genus

Grojnowski's Equivariant Elliptic cohomology:
$Ell_{\mathbb{T}}$: Finite $\mathbb{T}$-CW-complexes $\longrightarrow$ Sheaves of $\mathcal{O}_C$-algebras (hol fns)

C must be given as $\mathbb{C}/\Lambda$!

Features:

- Completion isomorphism $E_{\mathbb{T}}(X)^{\wedge}_0 \simeq E(X_{\mathbb{T}})$ elliptic coh of Borel constr.
- Pontryagin-Thom map $E_{\mathbb{T}}(X^T)^{-1} \longrightarrow E_{\mathbb{T}}(pt) = \mathcal{O}_C$ Thom space of tangent bundle over Borel const.
- Compatibility (invertible $E_T(X)$-module)

$$\begin{tikzcd} E_T(X^T)^{-1}(C) \arrow[r] \arrow[d] & E(X_T^{-\tau}) \arrow[d] & \ni \text{Witten Thom class} \arrow[d] \\ \mathbb{C} \simeq E_T(pt)(C) \arrow[r] & E(BT) & \ni W(M,V) \end{tikzcd}$$

Rosu: Want Thom class in Grojnowski equivariant ell cohomology --- which is what technical work of Bott–Taubes is allowing you to do...

Theorem (Ando) Let V be an $\mathbb{T}$-equivariant T-oriented virtual spin bundle over X s.t. $c_2(V_{\mathbb{T}})=0$ (Borel construction)
Then there is a canonical trivialization $\sigma(V)$ of $E_{\mathbb{T}}(X^V)$ → Thom space (global section of sheaf) whose image in $E(X_{\mathbb{T}}^{V_{\mathbb{T}}})$ is the equivariant Witten class.
The construction is natural & exponential in V

$\mathbb{T}$-oriented: choose orientation for $\overset{V^A}{\underset{X^A}{\downarrow}}$ for any closed subgroup of $\mathbb{T}$

$K(\mathbb{Z},3) \to BU\langle 6\rangle$ classifies SU bundles + trivialization of c_2 ... not just vanishing!
$\downarrow$
BSU Ando–Strickland: $Ell(K(\mathbb{Z},3))$ = set of Weil pairings on the elliptic curve! (~> dependence on lattice Λ)

Conceptual proof: K-theory $K(B\mathbb{T}) = \widehat{\mathbb{G}}_m \xleftarrow{\text{completion}} K_{S^1}(pt) = \mathbb{G}_m = \mathbb{Z}[\lambda,\lambda^{-1}]$

So should have $\widehat{C} = Ell(B\mathbb{T}) \longleftarrow E_{S^1}(pt) = C$

~~$Ell_{S^1}(pt) \cong C$~~ $Ell(B\mathbb{T}) = \widehat{C}$, $Ell(BT) = \check{T}\otimes\widehat{C}$ (torus; cocharacter lattice)

$Ell(BG) = (\check{T}\otimes\widehat{C})/W$

G-bundle on Borel construction $X_{\mathbb{T}} \longrightarrow BG \Rightarrow$

$$\begin{array}{ccc} \operatorname{Spec} Ell(X_{\mathbb{T}}) & \longrightarrow & (\check{T}\otimes\widehat{C})/W \\ \downarrow \text{completion} & & \downarrow \\ \operatorname{Spec} E_{\mathbb{T}}(X) & \xrightarrow{\text{"}f\text{"}} & (T^\vee\otimes C)/W \end{array}$$

$\mathcal{L}(V)\otimes\mathcal{I}(V)$, $\sigma(V)$

$\mathcal{L}$ line bundle $\otimes\mathcal{I}(V)$, σ

Sections of $\mathcal{L}^{\otimes k}$ = Reps of LG of level k
σ = character of basic rep of $\mathcal{L}$. with zeros

$BU\langle 6\rangle$ structure
$\updownarrow$
triv of $\mathcal{L}$

Conjecture: If V equiv spin bundle → map f exists
$\mathcal{I}(V) = f^*\mathcal{I}(\sigma) \cong E_{\mathbb{T}}(X^V)$ $\mathcal{I}(\sigma)$ ideal of zeros of σ.