M. Behrens – Isogenies of Elliptic Curves & the $K(2)$-local sphere

Topology 9/30/04

<u>Chromatic Filtration</u>

<u>Quillen's theorem</u>: $\mathrm{Spec}(MU_*) = \mathcal{M}_{FGL}$ moduli of formal group laws – formal groups with coord $G \xleftarrow{\sim} \hat{\mathbb{A}}^1$
– affine moduli space

$\mathrm{Spec}(MU_* MU) = \{\text{isomorphisms of formal } F \xrightarrow[f]{\sim} G\}$
$=$ changes of coordinates, acts on $\mathcal{M}_{FGL}$

$\Rightarrow$ quotient $\mathcal{M}_{FGL} // \mathrm{Spec}(MU_* MU) = \mathcal{M}_{FG}$
(remember tangent vector).

Stacky quotient $//$: topologically do with cobar construction: MU is $MU_* MU$ comodule $\Rightarrow$ cosimplicial object $\mathrm{Tot}\left(MU \rightrightarrows MU \wedge MU \rightrightarrows (MU \wedge MU) \hat{\wedge}_{MU} (MU \wedge MU) \cdots\right)$

$\| $ Bousfield

S the sphere spectrum $\therefore$

ie S : FGroups $::$ MU : FGLaws

Adams Novikov S.S. (Bousfield Kan)

$$H^*(\mathcal{M}_{FG}, \mathcal{O}) \xRightarrow[ANSS]{} \pi_* \mathrm{Tot}(MU \rightrightarrows \ldots) = \pi_* S$$

homotopy groups of spheres

Gross-Hopkins paper: useful table of analogies

<u>Work p-locally</u>

$\mathcal{M}_{FG} \supset V_n$ closed substack $= \{$all formal groups whose mod p reduction has height $\geq n$.$\}$

Height: look at p^{th} power map

$x \mapsto \underbrace{x +_G \cdots +_G x}_{p}$ $\quad [p](x) = (\,)x^{p^n} + \ldots \quad n = \text{height}.$

$V_n \supset V_{n+1} \supset \ldots\ldots$ decreasing closed subset

$U_n = \mathcal{M}_{FG} \setminus V_n$ increasing opens

$\ldots \subset U_n \subset U_{n+1} \subset \ldots$

$H^*(U_n) \underset{\text{ANSS}}{\Longrightarrow} \pi_*(L_n S)$ $\quad$ L_n-local sphere

(cohomology of structure sheaf)

$L_n = L_{E(n)}$ $\quad$ localize wrt $E(n)$ theory (Johnson-Wilson theory)

Chromatic tower

$$\begin{array}{c} \cdots \\ \downarrow \\ L_n S \\ \downarrow \\ L_{n-1} S \\ \downarrow \\ \cdots \end{array}$$

Bootstrap: Understand step $U_{n-1} \hookrightarrow U_n$: write pushout square $\quad$ homotopy pullback

$$\begin{tikzcd} U_{n-1} \cap \text{nbhd} \arrow[r, hook] \arrow[d] & U_{n-1} \arrow[d] \\ U_n \cap \text{Nbhd}(V_{n-1}) \arrow[r, hook] & U_n \end{tikzcd} \Longrightarrow \begin{tikzcd} L_{K(n)} S \arrow[d] & L_n S \arrow[l] \arrow[d] \\ L_{n-1} L_{K(n)} S & L_{n-1} S \arrow[l] \end{tikzcd}$$

(formal)

(could replace sphere by any finite complex)

So need to understand layers $L_{K(n)} S$ $\quad$ $K(n)$: Morava K-theory

Chromatic convergence theorem (Hopkins-Ravenel)

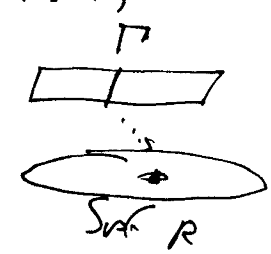
$S = \varprojlim L_n S$ $\quad$ (converges in strong sense)

homotopy inverse limit

Lubin-Tate moduli spaces

Fix ht n formal gp over char p perfect field k, (Γ, k). Fix R complete local ring

Look at formal groups over Spf R which restrict to Γ over closed pts



(Γ = Honda ht n fg / $\mathbb{F}_{p^n}$)

Identify any two deformations if have isoms which are identity on special fiber

$\Longrightarrow$ $\mathcal{M}_{LT} = \mathrm{Spf}((E_n)_*)$ $\quad$ affine formal scheme

E_n - Morava E-theory

classifying map

$$\begin{tikzcd} \mathcal{M}_{LT} \arrow[r] \arrow[d] \arrow[dr] & \mathcal{M}_{FG} \\ \mathcal{M}_{LT}/\mathbb{G}_n \arrow[r, Rightarrow] & U_n \cap \text{Nbhd } V_{n-1} \arrow[u] \end{tikzcd}$$

Morava: $\Rightarrow$ good approximation

$\mathbb{S}_n := \mathrm{Aut}(\Gamma)$ sees all extra isomorphisms — Morava Stabilizer group, $\mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}\, \mathbb{F}_{p^n}/\mathbb{F}_p$

Morava: $\mathcal{M}_{LT} // \mathbb{G}_n \longrightarrow U_n \cap \mathrm{Nbhd}\, V_{n-1}$
good approximation, in particular induces isomorphism on cohomology

$$\begin{array}{ccc} H^*(\mathcal{M}_{LT}//\mathbb{G}_n) & \xleftarrow{\sim} & H^*(U_n \cap \mathrm{Nbhd}\, V_{n-1}) \\ \| & & \Big\Downarrow \mathrm{ANSS} \\ H^*_c(\mathbb{G}_n, (E_n)_*) & & \pi_*(L_{K(n)} S) \end{array}$$

Note $(E_n)_* = \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]$
LHS very explicitly computable!
Devinatz-Goerss-Hopkins-Miller:
E_n is an E_∞ ring spectrum, $\mathbb{G}_n \circlearrowright E_n$
on the nose by E_∞ ring spectrum maps $\Rightarrow$
can take homotopy fixed points $E_n^{h\mathbb{G}_n} \simeq L_{K(n)} S$

So ANSS is just homotopy fixed pts s.s. here!

ht 1: look at moduli space (stack) of multiplicative group $\mathcal{M}_{mult} = \{\mathbb{G}_m\} \circlearrowleft \mathbb{Z}/2 \Rightarrow$ stack $B\mathbb{Z}/2$

Structure sheaf $\mathcal{O}_{mult}$ of E_∞ ring spectra
$\mathcal{O}_{mult}(\{\mathbb{G}_m\}) = KU$
Global sections $\mathcal{O}_{mult}(\mathcal{M}_{mult}) = (KU)^{h\mathbb{Z}/2} = KO$

Relation to $K(1)$-local sphere:

$$\begin{array}{ccc} \mathrm{End}(\mathbb{G}_m) & \hookrightarrow & \mathrm{End}(\widehat{\mathbb{G}}_m) \quad \text{ht } 1 \\ \cup & & \cup \quad (\text{Honda ht } 1) \\ \mathrm{End}_\ell(\mathbb{G}_m) & \longrightarrow & \mathrm{Aut}\, \widehat{\mathbb{G}}_m = \mathbb{Z}_p^\times = \mathbb{S}_1 \end{array}$$

(fix ℓ prime to p) — endos of degree a power of $\ell \to$ become invertible in formal completion

$\mathrm{End}_\ell\, \mathbb{G}_m$: maps $z \mapsto z^{\ell^k}$

$[\ell^k] \longrightarrow \ell^k \in \mathbb{Z}_p^\times = \mathbb{S}_1$

$$\begin{tikzcd}
\mathcal{M}_{mult} \arrow[rr] \arrow[d] \arrow[drr, dashed] & & \mathcal{M}_{FG} \\
\mathcal{M}_{mult}/\!/\mathrm{End}_\ell \arrow[rr, dashed, "\text{good approximation}"'] & & U_1 \cap \mathrm{Nbd}(V_0) \arrow[u]
\end{tikzcd}$$

what does $\mathcal{M}_{mult}/\!/\mathrm{End}_\ell$ mean, mod out by endos?

write simplicial stack $\mathcal{M}_{mult} \leftleftarrows \mathcal{M}_{mult}$

$$\underset{\text{objects}}{\{\mathbb{G}_m\}} \underset{\text{target}}{\overset{\text{source}}{\leftleftarrows}} \underset{\text{morphism}}{\{\mathbb{G}_m \xrightarrow{[\ell]} \mathbb{G}_m\}}$$

Apply $\mathcal{O}_{mult}$ to this diagram : get cosimplicial E_∞ ring spectrum

$$\mathrm{Tot}\left(KO \underset{1}{\overset{\psi^\ell}{\rightrightarrows}} KO \right) = J = L_{K(1)} S$$

Adams-Baird, Ravenel

Work $K(2)$ locally $(p=3)$ ($p \geq 5$ don't need tmf, $p=2$ hard ...)

Goerss-Henn-Mahowald-Rezk :

tower of spectra $TMF \longrightarrow TMF_0(2) \vee TMF \longrightarrow TMF_0(2) \vee \Sigma^{48} TMF_0^2 \longrightarrow \Sigma^{48} TMF \vee \Sigma^{48} TMF_0(2) \longrightarrow \Sigma^{48} TMF$

Composites are null homotopic, & Toda brackets vanish
$\Rightarrow$ refines to a spectrum, which is actually $L_{K(2)} S$

(conceptual approach to calculations of Shimomura-Wang)

Def A $\Gamma_0(\ell)$-structure on an elliptic curve C is a cyclic subgroup $H \subset C$ of order ℓ.

Replace $\mathcal{M}_{mult}$ by $\mathcal{M}_{ell}$ moduli of elliptic curves (nonsingular)
$\mathcal{O}_{mult} \rightsquigarrow \mathcal{O}_{ell}$ sheaf of E_∞ ring spectra

$\mathcal{O}_{ell}(\mathcal{M}_{ell}) = TMF$

$$\begin{tikzcd} \mathcal{M}_0(\ell) \arrow[d, "\text{étale}"] \\ \mathcal{M}_{ell} \end{tikzcd}$$

$\mathcal{M}_0(\ell)$ moduli of elliptic curves with $\Gamma_0(\ell)$ structure
$\Rightarrow$ can evaluate $\mathcal{O}_{ell}(\mathcal{M}_0(\ell)) = TMF_0(\ell)$

$\ell = 2$, $p = 3$

look at part $TMF \to \underset{TMF}{TMF_0(2) \vee} \to TMF_0(2)$ of resolution, and show how it dualizes to give other part.

C supersingular elliptic curve, $\hat{C}$ ht 2 formal group
$\operatorname{End} C \xrightarrow{\sim} \operatorname{End} \hat{C}$ both orders of quaternion algebras

after 3-completion get isomorphism here.

$$\begin{tikzcd} \operatorname{End} C \arrow[r] & \operatorname{End} \hat{C} \\ L := \operatorname{End}_\ell C \arrow[u, hook] \arrow[r, "\text{dense}"'] & \operatorname{Aut} \hat{C} = \mathbb{S}_2 \arrow[u, hook] \end{tikzcd}$$
monoid

$$\begin{tikzcd} \mathcal{M}_{ell} \arrow[rr] & & \mathcal{M}_{FG} \\ \mathcal{M}_{ell}^{ss} \arrow[u, hook] \arrow[d] \arrow[rr] & & U_2 \cap \operatorname{Nbhd}(V_1) \arrow[u] \\ \mathcal{M}_{ell}^{ss} /\!/ \operatorname{End}_\ell \arrow[rru, leftrightarrow] & & \end{tikzcd}$$

Good approximation? Ok, not great
not isomorphism

$\mathcal{M}$ = moduli of elliptic curves $\mathcal{M}^{ss}_{ell}$

Form simplicial stack:

$$\mathcal{M} \leftleftarrows \begin{array}{c}\mathcal{M}_0(\ell) \\ \sqcup \\ \mathcal{M}\end{array} \Lleftarrow \mathcal{M}_0(\ell)$$

Apply $\mathcal{O}_{ell}$: get cosimplicial E_∞ ring spectrum

$$\mathrm{Tot}\left(TMF \rightrightarrows \begin{array}{c} TMF_0(\ell) \\ \vee \\ TMF \end{array} \Rrightarrow TMF_0(\ell) \right) =: Q(\ell)$$

Maps: $$\underset{\text{ell curve}}{\{C\}} \leftleftarrows \begin{array}{c} \{ (C,H) \xrightarrow{\varphi_H} C/H \} \\ \sqcup \\ \{ C \xrightarrow{[\ell]} C \} \end{array} \Lleftarrow \left\{ \begin{tikzcd} (C,H) \arrow[rr, "{[\ell]}"] \arrow[dr, "\varphi_H"'] & & C \\ & C/H \arrow[ur, "\hat{\varphi}_H"'] & \end{tikzcd} \right\}$$

Conjecture [$K(2)$ Spanier-Whitehead dual D; $S \xrightarrow{\eta} Q(\ell)$ unit map, dually $\overset{S}{\overset{\shortparallel}{DS}} \leftarrow DQ(\ell)$]

then, $DQ(\ell) \longrightarrow S \longrightarrow Q(\ell)$ is a cofiber sequence

... Lagrangian for self-duality of $K(2)$ local sphere

Theorem True for $\ell = 2$, $p = 3$.

Theorem $DTMF = \Sigma^{44} TMF$

$44 + 4 = 48$

'' length of resolution