A. Beilinson – Chiral Algebras & Hecke Eigensheaves 4/24/96

(w/ Drinfeld) (Restrict to unramified case for simplicity)

$X$ compact curve, $G$, ${}^LG$, $Bun_G$ smooth alg. stack [not space of stable bundles]

$\rightsquigarrow$ notion of smooth $\mathcal{D}$-modules on $Bun_G$. Langlands: spectral decomposition of such $\mathcal{D}$-mods...

Spectral parameters: $LocSys_{{}^LG} \ni \phi \overset{?}{\dashrightarrow}$ subcategory of $\mathcal{M}(Bun_G)$ called $\mathcal{M}(Bun_G)_\phi$ — at least for $\phi$ irreducible.

$\phi$ reducible $\longrightarrow$ subcategory of derived category of $\mathcal{M}(Bun_G)$.

— space of Hecke $\phi$ eigensheaves.

$$Bun_G \xleftarrow{\pi_1} Hecke_x = \{(F_1, F_2, \nu)\} \xrightarrow{\pi_2} Bun_G$$

Fix $x \in X$ $\to$ Hecke correspondence

$F_1$, $F_2$ $G$ bundles, $\nu : F_1|_{X \setminus \{x\}} \xrightarrow{\sim} F_2|_{X \setminus \{x\}}$

$Hecke_x$ not alg. stack but union of a system of closed alg. substacks:

e.g. Trivial $F_0 \in Bun_G$, $\pi_1^{-1}(F_0)$ : choose local triv of $F_2$ near $x$, $\nu$ given by a function with values in $G$ on punctured disc. Change of triv $\leftrightarrow$ multiply by regular function on the disc, so $\pi_1^{-1}(F_0) = G(K_x)/G(O_x)$ affine Grassmannian of $G$, ind-scheme by bounding order of pole by something.

Nontrivial $F_0$ get twisted form of this

Hecke eigensheaves : get family of functors on the derived category of $\mathcal{M}(Bun_G)$ : $K \in \mathcal{M}(Hecke)$ gives functor $F_K : D\mathcal{M}(Bun_G) \to D\mathcal{M}(Bun_G)$,

$$M \longmapsto \pi_{2*}(K \otimes \pi_1^! M)$$

Special $K$'s to use : What exactly is $G(K_x)/G(O_x)$? consider left $G(O_x)$ action, orbits are true schemes labelled by set $G(O_x) \backslash G(K_x)/G(O_x)$.

Each fiber of Hecke $x$ looks like Grassmannian, twisting comes from left $G(O_x)$ action, equivariant $\mathcal{D}$-mods give well defined $\mathcal{D}$-mods on total space... use these as $K$'s.

$G(O_x)$-equiv $\mathcal{D}$-modules on $G(K_x)/G(O_x)$ : what is the set of orbits? Pick Cartan $H \subset G$,

$$\begin{array}{cc} & G(O_x) \backslash G(K_x)/G(O_x) \\ & \uparrow \\ \Gamma_H & H(K_x)/H(O_x) \end{array}$$

$\Gamma_H$ lattice of 1-parameter subgroups of $H$

$\Gamma_H \to G(\mathcal{O})\backslash G(K)/G(\mathcal{O})$ is onto, relations $W$: get bijection

$W\backslash\Gamma_H \to$ Now $\Gamma_H$ = characters of $H^\circ$ dual torus

so $W\backslash\Gamma_H$ is weights of $H^\circ$ modulo $W$, i.e.

isomorphism classes of ~~[illegible]~~ irreps of ${}^LG$, $W\backslash\Gamma_H = \underline{Irr({}^LG)}$..

Theorem (Lusztig) a). The category $\mathcal{P}$ of $\mathcal{D}$-mods smooth along our stratification is semisimple.

Irreducible objects – GM extensions of local systems on orbits..

but orbits are simply connected $\Rightarrow$

b). Irreducible objects $\leftrightarrow$ {orbits} $\leftrightarrow$ Irr ${}^LG$, as IC extensions of $\mathbb{C}_{orbit}$.

For symplectic reasons all orbits have same parity of dimension, but nontrivial $Ext^1$'s occur only for dimensions different by 1 $\Rightarrow$ semisimplicity.

Any $\mathcal{D}$-mod smooth along stratification $\Leftrightarrow$ equivariant.

$\mathcal{P} \xleftrightarrow{\sim} Rep\ {}^LG$ equivalent: sends irred objects to identified irred object.. canonical equivalence up to natural transformation. Ginzburg, Mirkovich-Vilonen, etc. etc: choose completely canonical such functor

$h: \mathcal{P} \xrightarrow{\sim} Rep\ {}^LG$.

So we have $Rep\ {}^LG \xrightarrow{\sim} \mathcal{P} \hookrightarrow \mathcal{M}(Hecke_x) \to$ Functors $(DM\ Bun G)$ $\circlearrowleft$

$V \mapsto F_{Vx}$

$h$ has remarkable monoidal property: $\otimes$ on $Rep\ {}^LG$

$\to$ composition of functors on $DM(Bun G)\circlearrowleft$ ...

Imprecise Def. $\underline{\text{Hecke eigensheaf wrt. point } x}$: $M \in DM(Bun_G)$ s.t.

- for any $V \in Rep\ {}^LG$ one has $F_V(M) \simeq V \otimes M$ weakly..
  - more precisely need fixed data of isoms $F_{Vx}(M) \xrightarrow{\sim} V\otimes M$ compatible with tensoring $V$'s..

$\to$ get vector space $V$ tensor our $\mathcal{D}$-mod $M$.

Variant: Assume we also have a ${}^LG$-torsor $\phi$ $\to$ may replace $V$ by $V$ twisted by $\phi$ : $F_{V_x}(M) \xrightarrow{\sim} V_\phi \otimes M$.

Now assume we vary the point $x$ : family over $Bun_G \times X$.

$$\begin{array}{ccc} & Hecke & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ Bun_G \times X & & Bun_G \end{array}$$

$\Rightarrow F_V : M(Bun_G) \to M(Bun_G \times X)$

May ask if for any $V$ $F_V(M) = \pi^!_{Bun_G}(M) \otimes V$

- too rigid unfortunately $\to$ true def:

Def of Hecke $\phi$-eigensheaf: Given $\phi$ ${}^LG$-loc sys twist $V \longleftrightarrow V_\phi$ flat bundle on $X$

$F_V(M) = M \boxtimes V_\phi$ canonically.

Spectral decomposition (conjecture): for any $\phi$ $\rightsquigarrow$ category of vector spaces.. Consider $\mathcal{O}$ mod on $Loc\,Sys_{{}^LG}$, should have canonical equivalence with $DM(Bun_G)$,

skyscraper $\rightsquigarrow V_\phi$.. known only for tori: Fourier-Mukai transform. - single irred object should exist.

How do we construct Hecke eigensheaves?

$\to$ Chiral Algebras.

Commutative chiral algebra: $A$ $\mathcal{O}_X$-algebra equipped with a connection

$\begin{array}{c} Spec\, A \\ \downarrow \\ X \end{array}$ $X$ scheme with connection $\to \mathcal{D}_X$-algebra:

system of nonlinear differential equations. e.g. $\begin{array}{c} Y \\ \downarrow \\ X \end{array}$ scheme

$\to$ scheme of jets of sections: $\begin{array}{c} JY \\ \downarrow \\ X \end{array}$. nonlinear version of induced $\mathcal{D}$-module

Solutions of equation - (flat) horizontal sections.. get actual affine scheme

Spec $H_\nabla(X, A)$ = horizontal sections.

Data of chiral algebra: $U \xrightarrow{j} X \times X \xleftarrow{\Delta} X$

$$\mu: \begin{array}{ccc} j_*(A \boxtimes A)|_U & \to & \Delta_*(A \otimes A) \text{ cokernel} \\ \uparrow & & \searrow \\ A \boxtimes A & \longrightarrow & \Delta_* A \end{array}$$

if product $\mu$ vanishes on $A \boxtimes A$, reduces to comm. assoc. algebra.

$H_\nabla(X,A)$ now just vector space (not algebra) but nonetheless completely canonical

Dual to $H_\nabla(X,A)$: replace $A$ by $O_X$ ..

$$\begin{array}{ccc} j_* O_U & \longrightarrow & \Delta_* O_U \\ \uparrow & & \uparrow \\ j_*(A\boxtimes A)|_U & \longrightarrow & \Delta_* A \end{array}$$

by residue map. a functional on $H_\nabla(X,A)$ is commutative diagram.

<u>From chiral algebras to Hecke eigensheaves</u> $A$ chiral alg. on $X$ $\longmapsto H_\nabla(X,A)$. Assume $A$ carries a $G(O_X)$ action. Then given a $G$ bundle $\leftrightarrow$ $G(O_X)$ torsor $\mathcal{F}\in Bun_G$ $\longmapsto A_{\mathcal F} \longmapsto H_\nabla(X,A_{\mathcal F})$ : get vector space for every point on $Bun_G$ ... in fact an $O$-module $\mathcal{H}(A)$.

How do we get a $\mathcal{D}$-module? -- assume we have also a $^LG$ action (constant group). Given $\phi \in LocSys_{^LG}$ $^LG$ torsor - twist by $\phi \to A_{\phi\mathcal F} \to H_\nabla(X, A_{\phi\mathcal F})$ $\to \mathcal{H}_\phi(A)$ $O$ mod on $Bun_G$.

For vacuum rep get just $\mathcal D$ on $Bun_G$, twisted by level line bundle.
If $A \supset Vac_k \to$ get canonical $\mathcal D$-mod structure
- integrable level can untwist to get $\mathcal D$ module.

To get <u>Hecke eigensheaf</u>: pick $k$ negative integral level
- don't know how to write chiral law on it by formulas, only the fibers - but can do it nicely abstractly.

Take fin dim rep of $^LG$ $\longrightarrow$ take $\mathcal D$-mod $h^{-1}(V)$, twist by line bundle on Grass corresp to level $\rightsquigarrow$ Hecke functor
let $x$ vary get $\mathcal D$-mod on curve which gives chiral algebra.
Start with trivial rep $\longrightarrow$ vacuum rep chiral algebra. Take regular rep. $^LG$ action comes as automorphisms of regular rep, $G(O_X)$ also comes
$\longrightarrow$ Hecke eigensheaves .. might be zero .. but if category is nonempty get them this way for high $k$.

Unlike normal Langlands have direct relation Galois $\rightsquigarrow$ automorphic forms