(see Langlands conference)

<u>Conjecture</u> (Langlands–R–Kottwitz) Assume G_{der} simply conn.
K_p parahoric. Then $\exists$ model s.t. (q-proj & flat)

$$\widetilde{Sh}_K(\bar{\mathbb{F}}_p) = \coprod_{\varphi \in \Phi} I_\varphi(\mathbb{Q}) \backslash X^p(\varphi)/K^p \times Y_p(\varphi)$$

$\Phi = \{$admissible morphisms of Galois grps $\mathcal{Q} \to \mathcal{G}_G\} / \sim$
quasi-motivic Galois grps

$(X^p(\varphi)/K^p) \simeq G(\mathbb{A}_f^p)/K^p$

$Y_p(\varphi)$: set with action of automorphism Φ.

Bijection equivariant wrt. Frob, $G(\mathbb{A}_f^p)$)

$\varphi \rightsquigarrow b(\varphi) \in \sigma$-conjugacy classes in $G(L)$

Kottwitz: functorial diagram

$$\begin{array}{ccc} G(L) & \xrightarrow{\tilde{\kappa}} & X^*(Z(G^\wedge)^I) \ \text{inertia} \\ \downarrow & & \downarrow \\ B(G) & \xrightarrow{\kappa} & X^*(Z(G^\wedge)^\Gamma) \ \text{Galois} \end{array}$$

. . . . . .

Next: do nonarchimedean uniformization of these sets!
formal scheme structure, coverings by simpler objects ... (stratification)
– analogous to Harder–Narasimhan stratification on moduli of vector bundles with Hodge/Newton polygons replacing H.–N. polygon.

<u>A. Beilinson</u>. Chiral Hecke Algebras 10/10/96

CFT
Class field theory ---- conformal field theory

1. Classical Langlands correspondence
(automorphic forms) : local fields here are function fields char p,
consider <u>global</u> unramified situation.
$X/\mathbb{F}_q$ smooth curve, F function field
$G/\mathbb{F}_q$ split reductive. ${}^LG / \bar{\mathbb{Q}}_\ell$

Unramified aut. forms : A = functions on $G(O_{\mathbb{A}}) \backslash G(\mathbb{A}) / G(F) \to \bar{\mathbb{Q}}_\ell$

A is a $\mathcal{H}_X$-module, $\mathcal{H}_X = \bigotimes_{x \in |X|} \mathcal{H}_x$

$\mathcal{H}_x = \bar{\mathbb{Q}}_\ell [G(\mathcal{O}_x) \backslash G(F_x) / G(\mathcal{O}_x)]$

$\wr\!\uparrow$

$W \backslash H(F_x) / H(\mathcal{O}_x)$

$\wr\!\uparrow$

$\mathcal{H}_x =$ fin. linear combos $[\mathrm{Irr}\ {}^LG.]$

pick Cartan
- lattice of one param groups = chars. for ${}^LG$

Satake identification $S_x : \mathcal{H}_x \to \mathcal{O}({}^LH)^W \leftarrow \mathcal{O}({}^LG/\mathrm{Ad})$

(canonical up to picking $q^{\frac{1}{2}}$ in $\bar{\mathbb{Q}}_\ell$)

$\leftrightarrow$ unramified characters of dual torus

conj. classes of ss elements

Langlands picture: set of Galois reps should correspond to this

$L : |\mathcal{SL}_{{}^LG}| \longrightarrow \mathrm{Spec}\, \mathcal{H}_X = \prod_{x \in |X|} {}^LG^{ss}/\mathrm{Adjoint}$

${}^LG$-local systems up to isom

$\ni \mathcal{F} \longmapsto \{$conjugacy classes of Fr_x^{ss} on fiber of $\mathcal{F}\}$

Langlands conjecture: $L(\mathcal{SL}_{{}^LG}) = \mathrm{Spec}$ of A

(eigenspaces of auto. forms)

General groups RHS has multiplicities, other problems.

$\alpha \in \mathrm{Spec}\, \mathcal{H} \to \dim A_\alpha$, $\# L^{-1}(\alpha)$ should coincide

Best way would be to concretely construct from local system $\mapsto$ aut. form.. but how to distinguish different ones with same eigenvalues? + other problems.

B. Geometric Langlands Correspondence — for automorphic sheaves.

Bun_G - smooth alg. stack = moduli of bundles, whose $\mathbb{F}_q$ points are space before.. $G(\mathbb{O}) \backslash G(\mathbb{A}) / G(F)$

Important functions here come from Weil or perverse sheaves.....

$K(\text{Weil sheaves}) \longrightarrow A$

$\mathcal{F} \longmapsto \mathrm{Tr}(Fr)$

$G(F_x)/G(\mathcal{O}_x) = \mathcal{G}r_x(\mathbb{F}_x)$ affine grassmanian
– an ind-scheme. Has left $G(\mathcal{O}_x)$ action $\Rightarrow$ stratification by fin dim orbits ...

Define $\widetilde{\mathcal{H}}_x = \{$ perverse sheaves on $\mathcal{G}r$, equivariant wrt $G(\mathcal{O}_x)$. $\}$
(category). $K(\widetilde{\mathcal{H}}_x) \longrightarrow \mathcal{H}_x$ normal Hecke algebra
as before, via $Tr(Fr)$ $\to$ function on double cosets.
(must pass from geometric sheaves $\mapsto$ Weil sheaves: requires again choice of $q^{\frac{1}{2}}$).

<u>Standard elements</u>: Irred sheaves, which are IC sheaves of orbits (simply connected) $\to$ elems of Hecke algebra.

<u>Lusztig</u>: Irred IC sheaf I_ν of orbit ν, V_ν irrep, then $S_x(Tr\,Fr\,(I_\nu)) = \chi_{V_\nu}$
character of V_ν (as ad-invariant function on G.)

[<u>Theorem</u> – construction (Drinfeld, Ginzburg, Mirkovic–Vilonen):
For $M \in \widetilde{\mathcal{H}}_x$ set $H(M) = \bigoplus H^i(Gr, M)$
Then $H(M)$ carries a canonical action of ${}^LG$.
$\Rightarrow$ ~~the~~ functor $\widetilde{S}_x : \widetilde{\mathcal{H}}_x \longrightarrow Rep({}^LG)$
which is an equivalence of categories (canonical!)]

$\widetilde{\mathcal{H}}_x$ is semisimple & we know its simple objects, so can do above noncanonically easily via simple objects $\to$ simple objects.
— One has $\widetilde{S}_x(I_\nu) \cong V_\nu$.

$x \in X \Rightarrow$ Hecke correspondence $\mathcal{H}ecke_x = \{(E, F, \nu : E|_{X \setminus x} \xrightarrow{\sim} F|_{X\setminus x}\}$

$$\begin{tikzcd} & \mathcal{H}ecke_x \arrow[dl] \arrow[dr] & \\ Bun_G & & Bun_G \end{tikzcd}$$

fiber is twisted form of affine Grass. wrt $G(\mathcal{O}_x)$ action.
$\to$ our sheaves live on every twisted form by equivariance

$\Rightarrow$ get functor $Rep\,{}^LG \longrightarrow Perv(\mathcal{H}ecke_x)$
$V \longmapsto \mathcal{M}_V$

Definition A perverse sheaf P on Bun_G is a Hecke eigensheaf at x for $\mathcal{F}_x$ - ${}^LG$-torsor at x (eigenvalue)
if for any $V \in Rep\, {}^LG$ $\quad M_{V*}(P) \underset{\text{given}}{\cong} V_{\mathcal{F}_x} \otimes P$ $\longleftarrow$ twist
(M_{V*} is correspondence on $\mathcal{D}$(perv) by pullback & twist)

Globally: $\mathcal{F} \in \mathcal{LS}_{{}^LG}$ - ${}^LG$ loc sys on X
$M_V(P) = V_{\mathcal{F}} \boxtimes P$.
$\Longrightarrow$ corresponding aut. form is Hecke-eigenform.
Really need full local system on X not just ${}^LG$ torsor at each point ... ~~Hope is~~ given irred loc sys $\longrightarrow$ unique corresponding Hecke eigensheaf, and corresp forms give basis in space of cusp forms!!

Now we can replace $\mathbb{F}_q$ by $\mathbb{C}$ .. situation is better
- replace Perv. sh. by $\mathcal{D}$-modules, much more friendly & flexible

Reckless Conjecture $\quad D^b(\mathcal{O}\text{mod on } \mathcal{LS}_{{}^LG}) \xrightarrow[\sim]{\text{canonical}} D^b(\mathcal{D}\text{-mod on } Bun_G)$ natural wrt tensoring
- skyscrapers $\to$ Hecke eigensheaves
~ nonabelian Fourier transform .. can't be stated only with perverse sheaves: need $\mathcal{D}$-mods.

We want to find "integral kernel" for this
- we'll just give local version of this depending on a level for KM algebra - gets larger with level, hopefully approaching full correspondence.

$$\begin{array}{ccc} & \mathcal{D}\text{mod} \boxtimes \mathcal{O}\text{mod} & \\ \swarrow & & \searrow \\ Bun_G & & \mathcal{LS}_{{}^LG} \end{array}$$

(1) CFT - chiral algebras
X curve. A left $\mathcal{D}$-mod on X.
Def. A chiral alg. structure on A is the following:

to any finite set $(\neq\emptyset)$ I get $\mathcal{D}$-mod $A^{(I)}$ on X^I, flat as $\mathcal{O}$-mod
+ compatibilities : $A^{(1)} = A$

$X \overset{\Delta}{\hookrightarrow} X^2 \hookleftarrow U$: $A^{(2)}|_X$: $\Delta^* A^{(2)} = A$
$A^{(2)}$ $\quad A^{(2)}|_U = A \boxtimes A|_U$.

On X^I : $S \subset X$ finite set , $A_{\otimes S} = \bigotimes_{s \in S} A_s$

$(x_i) \in X^I$, $A^{(I)}_{(x_i)} = A_{\otimes\{x_i\}}$ - <u>without</u> multiplicities.

Ex. put $\mathfrak{g}$ over any point $\mathfrak{g} \otimes K_x \to$ Vacuum rep
family of vacuum reps form chiral algebra.

<u>Chiral (co)homology</u> : $X \rightsquigarrow R(X)$ space of finite nonempty subsets of X, with natural topology

<u>Exer.</u> (i) for X a curve $R(X)$ has no local functions at all...
(so not alg. variety)
(ii) $R(X)$ is contractible. $X \hookrightarrow R(X)$.

A chiral algebra defines a sheaf on $R(X)$ ("$\mathcal{D}$-mod")

<u>Def</u> $H_i^{ch}(A) = H_i^{dR}(R(X), A^{R}) = \varinjlim H_{dR}^{2n-i}(X^I, A^{(I)})$
H_0 = conformal blocks.

Assume A carries an action of ${}^L G \times G(\mathcal{O}_x)$ (sheaf)
$\Rightarrow$ $\mathcal{O}$-mod on $Bun_G \times LS_{{}^L G}$:
$K(A)|_{(\mathcal{F}, R)} = H_i^{ch}(A_{(\mathcal{F},R)})$
${}^L G$ loc sys $\quad \to G$ bundle

If $A \supset Vac_k\, \mathfrak{g}$ compatible with $G(\mathcal{O}_x)$ action, fixed by ${}^L G$
$\Longrightarrow$ $\mathcal{D}$-mod structure on Bun_G.

(E) <u>Chiral Hecke Algebras</u>

<u>Def</u> A chiral ind-scheme $/X$ is an ind-scheme $\mathcal{G}/X$ equipped with a connection along X +
on any $X^I \Rightarrow \mathcal{G}^{(I)}$ with connection & same compatibilities

$(\mathcal{G}^{(I)}_{(x_i)} = \prod_{s \in S} \mathcal{G}_s)$. (no multiplicities).

<u>Example</u> Affine Grassmannians/X have chiral structure:
$\mathcal{G}r_x$ = moduli of G-bundles trivialized on $X \setminus x$ => family with connection. $\rightsquigarrow$ $\mathcal{G}r_{(S)}$ local wrt points
Level - comes from line bundle on $\mathcal{G}r$
Chiral algebra: take δ-functions along some universal section of $\mathcal{G}r$ - e.g. trivial bundle => Vacuum rep. (of any level, by twist)

Note $\mathcal{D}$-mod on $\mathcal{G}$ gives $\mathcal{D}$-mod on X by pushforward
<u>Theorem</u> For any rep of ${}^LG$ we get a $\mathcal{D}$-mod $\mathcal{M}_V$ on $\mathcal{G}r$, take its $p_*(\mathcal{M}_V \otimes \mathcal{L}^k)$ => $\mathcal{D}$-mod on X.
For a <u>ring-object</u> (commutative) in Rep ${}^LG$ => this is a chiral algebra. e.g. trivial ⇒ Vacuum VOA.
Regular rep ⇒ =: chiral Hecke algebra.
Carries natural ${}^LG \times G(\mathcal{O})$ structure, => produces (as above) Hecke eigensheaves.
(can check - residues at ∞ => Eisenstein series
can also take Whittaker coefficients etc. - all local VOA computations - problem is to compute quantum DS reduction - only abelian, Heisenberg case is known.. nonabelian case?)

10/11

<u>T. Spencer</u> - A path integral view of 2-d percolation

Langlands, Pouliot, St. Aubin BAMS <u>30</u> 1-61

Percolation - seeping of liquid through porous medium
<u>Independent</u> bond percolation - fluid moves on edges of a square lattice - mark blue bonds with equal probability p, see if a connection is made from left side to right