A. Beilinson - Quantization of Hitchin's Hamiltonians & Hecke Eigensheaves $\quad$ 10/2/95
(w. Drinfeld) $\quad$ Conjecture (D.) - construct D-mods on moduli from opers using CFT, which are Hecke eigensheaves...

<u>Local picture</u> $\quad$ Feigin-Frenkel - strange from geometric P.O.V., corollary of special duality of quantum W-algebras, specialized at critical level. Here give geometric picture only for classical part..

$\mathfrak{g}$ semisimple, $K = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$, central exts of $\mathfrak{g} \otimes K$ labelled by symmetric forms on $\mathfrak{g}$ $\quad c \Rightarrow U_c(\mathfrak{g} \otimes K) \to \hat{U}_c(\mathfrak{g} \otimes K)$. $\boxed{\text{Fix } B \subset G}$

$c \leftrightarrow -\frac{1}{2}$ Killing critical level ($\mathfrak{g}$ simple) - huge center:

So take $c = -\frac{1}{2}Tr(ad, ad)$, $\mathfrak{Z}$ = center.

In $gr\, \hat{U}_c = Sym\, \mathfrak{g} \otimes K$ = poly. on $\mathfrak{g}^* \otimes \omega_K$ $\quad$ ($\omega_K = K^*$ canonically).

$\to$ Initial estimate $\quad gr\, \mathfrak{Z} \subset (\;)^{\mathfrak{g} \otimes K}$ = poly. functions on $\mathfrak{h}^*/W \cdot \omega_K$

• using regular orbits for corresponding group $Map(D, G)$

<u>FF</u>: the above embedding is iso..

$\mathfrak{Z}$ is deformation of $gr\, \mathfrak{Z}$

Consider $Vac_c = Ind_{\mathfrak{g} \otimes \mathcal{O}}^{\mathfrak{g} \otimes K} \mathbb{C}_c$, $\quad \hat{U}_c$ acts on Vac_c, so does $\mathfrak{Z}$ - huge ideal kills this vacuum $\Rightarrow$ ideal $I \subset \mathfrak{Z}$ annihilating Vac_c.

<u>F-F</u>: image of $\mathfrak{Z}$ in $End(Vac_c)$ is all: $\mathfrak{Z}/I \xrightarrow{\sim} End_{\hat{\mathfrak{g}}} V_c$

$\mathfrak{z} = \mathfrak{Z}/I$, get $gr\, \mathfrak{z}$ = functions on $\mathfrak{h}^*/W \cdot \omega_{\mathcal{O}}$ in $\mathfrak{Z}$

<u>Opers</u> $\quad X$ - curve, a G-oper over X is a triple $(\mathcal{F}_G, \mathcal{F}_B, \nabla)$

$\mathcal{F}_G$ G-torsor over X, ∇ connection on this, $\mathcal{F}_B$ B-structure on $\mathcal{F}_G$ (reduction to B), with Griffiths transversality in "strict sense"...

Standard way of going from system of order-1 eqs to order n eq.

Very rigid object - no nontrivial auts (besides center (G)) - i.e. $\mathfrak{g}$-opers are rigid.

<u>Description</u>: $\;sl_2 \hookrightarrow \mathfrak{g}$ - may induce sl_2-opers to $\mathfrak{g}$-opers
principal embeddings $\;b_{sl_2} \hookrightarrow b$

$\mathfrak{g}$-opers for non-$\mathfrak{gl}_n$ - have obstruction to ∇ preserving flag - $(\mathfrak{g}/b)_{\mathcal{F}_B} \otimes \omega_X$ carries obstruction, we want our obstruction to fill $(\mathfrak{g}_{-1}/b)_{\mathcal{F}_B} \otimes \omega_X$

Want to express every $\mathfrak{g}$ oper as induced sl_2 oper + standard correction: $\quad$ take $L \in sl_2$ generator of opposite nilp $\;n_{sl_2} \subset b_{sl_2}$,
get $\mathfrak{g}^{n_{sl_2}} = V_{\mathfrak{g}} \subset b$ - invariants $\quad \mathfrak{g}^{n_{sl_2}}$ subspace of dim = rank of the algebra.

So add to each induced a correction in $V_{\mathfrak{g}}$:

<u>Fact</u> $\quad \Gamma(X, (V_{\mathfrak{g}})_{B_{sl_2}} \omega_X) \xrightarrow{\sim} \mathfrak{g}$-opers $\quad$ ($V_{\mathfrak{g}}$ twisted bundle - twisted by B_{sl_2}) - $V_{\mathfrak{g}}$ is B_{sl_2}-module.

So start with a B_{sl_2}-oper, get this map & it's an isomorphism.
- induce, then change the connection by adding this $V_{\mathfrak{g}}$ potential...

Local situation – functions on moduli space of opers have canonical filtration, whose assoc. graded is gr $\mathfrak{Z}$ ..

Standard projection (Kostant ⊗) $V_{\mathfrak{g}} \twoheadrightarrow \mathfrak{h}/W_{\mathfrak{g}}$

$v \mapsto$ class of $L+v$ (L opposite nilpotent to n_{sl_2}).

So up to translation, $\mathfrak{g}$-opers are the same as elements of $\mathfrak{h}/W \cdot \omega_K$ (on $\mathbb{C}^*$ or formal punctured disc.).

Now Spec gr $\mathfrak{Z}_{\mathfrak{g}} = \mathfrak{h}^*/W\,\omega_K$

$\Rightarrow$ Moduli of ${}^L\mathfrak{g}$-opers on Spec K or $\mathcal{O}$ is a deformation of Spec gr $\mathfrak{Z}_{\mathfrak{g}}$ or gr $Z_{\mathfrak{g}}$.

Th. (F–F) $\exists$ a canonical isomorphism between these deformations!

Spec $\mathfrak{Z}_{\mathfrak{g}}$ = moduli of ${}^L\mathfrak{g}$ opers on Spec K

" $Z_{\mathfrak{g}}$ = " " " " " Spec $\mathcal{O}$.

Canonical – symmetries of Spec $\mathcal{O}$ act on whole picture, and the isomorphism is Aut $\mathcal{O}$-equivariant.

We will define the arrow Spec $Z_{\mathfrak{g}} \to {}^L\mathfrak{g}$-op on Spec $\mathcal{O}$

Part 2. Satake equivalence (geometric interp of $\mathfrak{Z}$) – due here to Drinfeld, Ginzburg, Mircovic–Vilonen – extending Satake, Lusztig geometrically.

$G \uparrow H$ ${}^LG \uparrow H^\vee$ Usual Satake – G split semisimple$/\mathbb{Q}$, take Hecke(G) = measures with c. supp on $G(\mathbb{Q}_p)$, $G(\mathbb{Z}_p)$ bi-invariant, perfect ring (commutative) wrt convolution.

Satake identifies this with (Functions on $H^\vee$)^W = Repn. ring of ${}^LG$ (repn $\Rightarrow$ character), similar to Harish-Chandra iso – consider unramified principal series of $G(\mathbb{Q}_p)$, character is elem of $H^\vee$, canonical vacuum vector, elt of Hecke $\to$ function on $H^\vee$ . . . principal $\to$ Verma get Harish-Chandra.

Replace algebras with categories, rep ring $\Rightarrow$ ⊗ category, Hecke repn $\Rightarrow$ perverse on double coset sheaves : $G(\mathbb{Q}_p)$ ind-scheme over $\mathbb{F}_p$ ....

Replace $G(\mathbb{Q}_p)$ by $G(K)$ – $\mathbb{C}$ points of a group ind-scheme$/\mathbb{C}$ $G(\mathcal{O})$ pro-alg group, take affine Grass $= G(K)/G(\mathcal{O})$

$\mathcal{P}_G$ = category of $G(\mathcal{O})$-equiv perverse sheaves on $G(K)/G(\mathcal{O})$

– irred ones numbered by orbits, there are no nontrivial exts between these, so = p.sh. smooth along $G(\mathcal{O})$-orbits

This is a tensor category (modelled on convolutions of p.sh.)

– monoidal ~~tensor~~ category...

From Lusztig thm $\Rightarrow$ semisimple cat, irreds $\leftrightarrow$ orbits, numbered by high weights of irred reps

Fiber functor : $\oplus H^i(G(K)/G(O), \bullet) : \mathcal{P}_G \to Vect$
(just H_{DR} on D-mods.) $\Rightarrow$ group of automorphisms of H_{DR}
is <u>canonically</u> ${}^LG$. ($\mathcal{P}_G$ commutative not as VOA...)
= Langlands dual group, via a Satake isomorphism.

What is the distinguished Borel of ${}^LG$ in this picture?
$\Leftrightarrow$ finding a <u>line</u> of principal vectors for each ${}^LG$-rep
But irreds are IC sheaves of some orbit — lowest IC is $\mathbb{C}$, so gives a line! $\Leftrightarrow$ Borel ${}^LB < {}^LG$ canonical.

(Twisted) D-modules on $G(K)/G(O)$: our perverse sheaves sit on fin-dim part, carry through Riemann-Hilbert.. What is D-mod on ind scheme, especially here limit of singular schemes?

Embed singular $\to$ smooth, define via Kashiwara equivalence.
What are sections of D-mod (right)? $Y \hookrightarrow V$ Y singular V smooth
$\mathcal{M}_{mod}(Y) := \mathcal{M}^r(V)_Y$, gives canonical defn on singular manifolds (independent of ...) for <u>right</u> D-modules
Canonically these are crystals (without embedding) ...
sections defined inductively via support on limits etc etc etc.

Half-forms on $G(K)/G(O)$ λ - defined canonically as follows: <u>Fix</u>
$\mathcal{L}_O = \omega_O^{1/2}$ on Spec O - unique up to <u>sign</u> .

Fix ( ) nondegen scalar product on $\mathfrak{g}$
Space $\mathfrak{g}(K) \otimes \mathcal{L}_O$ carries canonical scalar product : twist ( ) on K by $\mathcal{L}_O$, get $\mathcal{L}_O \otimes \mathcal{L}_O \to$ 1-forms on disc, take Res ( )
$\mathfrak{g}(O) \otimes \mathcal{L}_O$ is Lagrangian wrt this form — so every pt in Grass gives Lagrangian subspace

Now define fiber λ_g of λ : $\lambda_g = \det\left(Ad_g(\mathfrak{g}(O))^{\otimes \mathcal{L}} / Ad_g(\mathfrak{g}(O)) \cap \mathfrak{g}(O)^{\otimes \mathcal{L}} \right)$
- relative Pfaffian of two subspaces, depends continuously on g even though inside spaces jump...
- makes perfect sense on each orbit $\Rightarrow$ ($\to$ finite dimensional)

$\boxed{\text{Fact}}$ The restriction of λ to any orbit C coincides with $\omega_C \otimes$ 1-dim'l space rewrite λ as the opposite det :
$\det^*(\mathfrak{g}(O) \otimes \mathcal{L} / C)$)

$\Rightarrow$ $IC_C \leftrightarrow M_C$, right D-mod corresp to IC of orbit C.
Semisimplicity theorem $\Rightarrow$ IC sheaf $M_C = H^0(j_{C*}(\omega_C)) \supset j_{C!*}\omega_C$
(Lusztig). (as sheaves)

j_C inclusion

Now twist by λ :

$M_c \otimes \lambda^{-1} \supset i_{c*}\mathcal{O} \supset \mathbb{C}$ - canonical line of sections, δ-functions corresp to orbit ! $\delta_c \in \Gamma(M_c \otimes \lambda^{-1})$

Theorem For any $M \in \mathcal{P}_G$ the higher (plain) cohomologies $H^i(G(\mathcal{O})\backslash G(\mathcal{K}), M \otimes \lambda^{-1})$ vanish for $i > 0$, and $\Gamma(M \otimes \lambda^{-1})$ - [$\hat{\mathfrak{g}}_{\kappa}$ acts here, critical level algebra] is isomorphic to a direct sum of fin. many copies of Vac_c as a $\bar{V}_c$-module · [delta-functions just give Vac_c, and any other M will just be a direct sum of such...] [# of copies $\leftrightarrow$ dim of deRham]

Birth of opers $\phi(M) = Hom_{\bar{V}_c}(Vac_c, \Gamma(M \otimes \lambda^{-1}))$ free Z-module of finite type, Z being again $End_{\bar{V}_c}(Vac_c)$

Proposition $\phi: \mathcal{P}_G \to$ ~~locally~~ free Z-modules is a fiber functor. (everything strictly commutative)

Now any such fiber functor comes from a torsor — unique up to twisting by torsor which is Hom(fiber, standard fiber = de Rham cohomology).

! Thus we defined a canonical ${}^LG$-bundle $\mathcal{F}_{{}^LG}$ on Spec Z - $\phi(M) = H_{DR}(M)_{\mathcal{F}_{{}^LG}}$.

Extra structure i) There is a canonical reduction $\mathcal{F}_{{}^LB}$ of $\mathcal{F}_{{}^LG}$ to ${}^LB$ - to do this we need to describe line in any $\mathcal{F}_{{}^LG}$-twisted reps - this will come from the $\delta_c \in \Gamma(M_c \otimes \lambda^{-1})$.

ii) Aut$(\mathcal{O})$ acts on this picture (ind-group scheme - contains integrable part - Auts preserving a point — + formal part shifting the pt.). $\mathcal{F}_{{}^LG}$ is Aut-$\mathcal{O}$ equivariant. Now Aut$(\mathcal{O})$ shift orbits C if they don't fix point - formal part doesn't preserve δ_c — so the reduction $\mathcal{F}_{{}^LB}$ is only Aut$^0(\mathcal{O})$-equivariant.

Our aim is to send Spec $Z \to$ moduli of ${}^LG$ opers on Spec $\mathcal{O}$:

$z \in$ Spec Z, Aut$(\mathcal{O}) \to$ Spec Z as orbit $g \mapsto gz$.

Pull back $\mathcal{F}_{{}^LG}, \mathcal{F}_{{}^LB} \Rightarrow \mathcal{F}_{{}^LG}, \mathcal{F}_{{}^LB}$ on Aut $\mathcal{O}$,

$\mathcal{F}_{{}^LG}$ is equiv wrt Aut $\mathcal{O}$ - so gives constant (trivial) ${}^LG$-torsor,

$\mathcal{F}_{{}^LB}$ constant wrt translations Aut$^0(\mathcal{O})$ (Aut$^0(\mathcal{O})$-equivariant)

— IN OTHER WORDS, $(\mathcal{F}_{{}^LG}, \mathcal{F}_{{}^LB})$ come from the quotient Aut$^0(\mathcal{O}) \backslash$ Aut$(\mathcal{O})$ = Spec $\mathcal{O}$ —

constant ${}^LG$ bundle on Spec $\mathcal{O} \leftrightarrow$ bundle with connection...

Prop This is a ${}^LG$ oper on Spec $\mathcal{O}$.

Explanation of Commutativity :

Different def. of $\otimes$ on $\mathcal{P}_G$ (which will agree with convolution):
perverse sheaves on $G(O)\backslash G(K)/G(O)$ indep of choice of disc — aut disc equivariant. Now take X a curve. Motivation (Drinfeld) for commutativity: Hecke operators at different points commute obviously.. try to take limit of that picture

F_X bundle on X, fiber $F_x = G(K_x)/G(O_x)$ — Grassmannian bundle — (with canonical connection)
get perverse sheaf on total space F_X as family.

Now take a finite set of points S.. First note $F_x = G(X \setminus \{x\})/G(X)$ for X affine this doesn't change on formal neighborhood $\Rightarrow$ connection. Distinguished horizontal section 1 — only horiz. section (connection doesn't preserve stratification)

$S \subset X$ Set $F_S = \prod_{x\in S} F_x = G(X \setminus S)/G(X)$

$\Rightarrow$ bundle F_{X^n} on X^n, with fiber $F_{(x_1,\dots,x_n)} := F_{\{x_1,\dots,x_n\}}$
— gives true, formally smooth bundle.. Over diagonal this bundle is not product but only one copy of F: they convolve on the diagonal.

Canonical flat connection... Now define product

$$\begin{tikzcd} F_{X_0} \arrow[r, hook] \arrow[d] & F_{X^2} & F_U = (F_X \times F_X)|_U \arrow[l, hook'] \arrow[d] \\ X \arrow[r, hook, "\Delta"] & X^2 & U \arrow[l, hook'] \end{tikzcd}$$

, U complement of diagonal

Now say $M_1, M_2 \in \mathcal{P}_G$. Consider $M_1 \boxtimes M_2 \Rightarrow$ per. sheaf over U (on $F_X \times F_X|_U$): Now extend, get $j_{!*}(M_1 \boxtimes M_2|_U)$, perverse sheaf on X, whose pullback $\Delta^* j_{!*}(M_1 \boxtimes M_2|_U)$ is per. sheaf on $X \Longrightarrow M_1 \otimes M_2$, obvious commutativity & assoc (like Lusztig's interp of Springer corresp.)

This agrees with convolution:
(small resolution): point in $\tilde{F}_{X^2}$ is in $G(X \setminus \{x,y\})/G$

$$\begin{tikzcd} & \tilde{F}_{X^2} \arrow[dl, "\text{small projection}"'] & \\ F & & F_U \arrow[ll, hook'] \arrow[ul] \end{tikzcd}$$

Consider $\mathcal{F}^0 \to \mathcal{F}^1$ sheaves on X, $\mathcal{F}^0$ trivialized, and these two isom on $X \setminus \{x\}$ — classes of such data are same as $G(X \setminus \{x\})/G(X)$.
Now take a pair of pts x, y, consider triple $(\mathcal{F}^0 \xrightarrow[\text{off } x]{} \mathcal{F}^1 \xrightarrow[\text{off } y]{} \mathcal{F}^2)$ isoms
These give ind sch on $X \times X$, which is $\tilde{F}_{X^2}$, with projection to F which is composition $\mathcal{F}^0 \to \mathcal{F}^2$.

Fiber — way to represent elem of $G(K)/G(O)$ as composition — i.e. we get convolution .. twisted (by $\mathcal{F}^1$) version of $F_X \times F_X$ ...
$j_{!*}(M_1 \boxtimes M_2)$ to $\tilde{F}$ pushes forward to one on F_{X^2} by smallness..

This is geometric formulation of OPE of Vac: Vac is δ-functions at a pnt x — sections of sheaves of δ function — given two sections of Vac assign a single section over X^2 — i.e. OPE..

Hitchin system - global analog on a curve of the center gives Hitchin hamiltonians, passing from quantum objects to global get quantization of Hitchin system as D-modules - which are automatically Hecke eigensheaves from construction