IAS

4/9/97

A. Beilinson – Hecke Eigensheaf Property

Assume for now $G$ simply connected

$A^{glob}$ = ring of functions on space of ${}^LG$-opers.

We have $A^{glob} \longrightarrow \Gamma(Bun_G, \mathcal{D}')$

$\phi$ ${}^LG$ oper $\Rightarrow$ max ideal in $A^{glob}$ $\Rightarrow$ $m_\phi \subset A^{glob}$

$\to \mathcal{D}'/\mathcal{D}' m_\phi$ with singular support zero-fiber of Hitchin map.

Untwist : show $\omega_{Bun_G}^{\frac{1}{2}}$ exists – it will automatically be unique for $G$ simply connected.

<u>Claim</u> $\exists\ \omega_{Bun_G}^{\frac{1}{2}}$.

<u>Sketch of construction</u> : Fiber of canonical bundle

$\omega_{\mathcal{F}} = \det R\Gamma(X, \mathfrak{g}_{\mathcal{F}})$ , $\omega^{\frac{1}{2}}$ will be Pfaffian

– need symmetric nondegen pairing with value in $\omega$ –

use symmetric nondegen form $(\,,\,)$ on $\mathfrak{g}$ (<u>invariant</u>)

$\Rightarrow \mathfrak{g}_{\mathcal{F}} \times \mathfrak{g}_{\mathcal{F}} \longrightarrow \mathcal{O}_X$

Pick theta-characteristic $\omega_X^{\frac{1}{2}}$ on $X$ $\Rightarrow$

$\mathfrak{g}_{\mathcal{F}}\omega_X^{\frac{1}{2}} \times \mathfrak{g}_{\mathcal{F}}\omega_X^{\frac{1}{2}} \longrightarrow \omega_X$ $\Rightarrow$ get Pfaffian

$Pf(X, \mathfrak{g}_{\mathcal{F}}\omega_X^{\frac{1}{2}})$

May identify $\det R\Gamma(X, \mathfrak{g}_{\mathcal{F}}\omega_X^{\frac{1}{2}}) = \det R\Gamma(X, \mathfrak{g}_{\mathcal{F}})$ :

represent $\omega_X^{\frac{1}{2}}$ as $\mathcal{O}$(divisor w/simple zeros, poles)

so difference between these two dets is det of fibers $\mathfrak{g}_{\mathcal{F}}$ at these points of zeros & poles – but this is indep of $\mathcal{F}$ – so all is determined up to twist by constant line.

$\Longrightarrow$ left $\mathcal{D}$-modules $\omega_{Bun_G}^{\frac{1}{2}}\mathcal{D}'$ on $Bun_G$

Morphism $A^{glob} \to \Gamma(Bun_G, \mathcal{D}')$ gives right action on this.

We get family of left $\mathcal{D}$-mods (holonomic) parametrized by opers, $\phi \longmapsto M_\phi$

<u>Theorem</u>! This is a Hecke $\phi$-eigensheaf.

- Introduce Hecke stack $\mathcal{H}ecke$ :

[redacted]

$$\begin{array}{c}\mathcal{H}ecke\\ \downarrow\\ Bun_G \times Bun_G \times X\end{array}$$

$\mathbb{C}$-points of $\mathcal{H}ecke$ are $\{(\mathcal{F}_1, \mathcal{F}_2, x, \alpha)\}$,
$\mathcal{F}_i$ $G$-bundles on $X$, $x \in X$, $\alpha : \mathcal{F}_1|_{X \setminus x} \xrightarrow{\sim} \mathcal{F}_2|_{X \setminus x}$.
This is a formally smooth ind-algebraic stack.
If we fix $x \in X$, form a groupoid by composition of $\alpha$'s $\Rightarrow$ $X$-family of groupoids on $Bun_G$.
Exercise Define a connection along $X$, canonical, with projection to $Bun_G \times Bun_G \times X$ horizontal..

Fiber $\mathcal{H}ecke_{(\mathcal{F}_1, x)}$ : twisted form of affine Grassmannian
$\mathcal{GR}_x \doteq G(K_x)/G(\mathcal{O}_x)$ naturally ind scheme;
formally smooth, ind-proper ind-scheme
$\mathcal{F}_1(\mathcal{O}_x)$ is $G(\mathcal{O}_x)$-torsor, & fiber $\mathcal{H}ecke_{(\mathcal{F}_1, x)}$ is $\mathcal{F}_1(\mathcal{O}_x)$-twist of $\mathcal{GR}_x$:
$\gamma_1 \in \mathcal{F}_1(\mathcal{O}_x)$, $\gamma_2$ local trivialization of $\mathcal{F}_2$
so we send $\gamma_1$ to $\alpha(\gamma_1)/\gamma_2$ modulo right multiplication by $G(\mathcal{O}_x)$.

$\mathcal{GR}_x$ has stratification by fin-dim varieties – $G(\mathcal{O}_x)$ orbits,
$\mathcal{GR}_x$ is inductive limit of their (singular, reduced) closures.
This stratification is labeled by positive weights
$P^+({}^LG) \ni \chi \leftrightarrow$ one param subgroup of Cartan of $G$ $\chi^\vee : G_m \to G$. $\chi^\vee(t_x) \in G(K_x)$
$\Rightarrow$ orbit $Orb_\chi = G(\mathcal{O}_x) \cdot \chi^\vee(t_x) \cdot G(\mathcal{O}_x)$.
These orbits are simply connected and even-dimensional, embeddings very much not affine.
$I_\chi$ – "intersection cohomology" $\mathcal{D}$-module on $Orb_\chi$. (note – $\mathcal{D}$-mod on singular scheme.. define by Kashiwara theorem).

In fact stratifications of the fibers glue together to give stratification (with same indices) of $\mathcal{H}ecke$, strata $\mathcal{H}ecke_\chi$

smooth fin-dim projection - loc triv fibration

$\mathcal{H}ecke_{\chi} \xrightarrow{i_\chi} \mathcal{H}ecke$

$\downarrow p_{1\chi}$

$Bun_G$ Hecke functors $T_\chi$ : $M$ $\mathcal{D}$-mod on $Bun_G$

$\Rightarrow$ $p_{1\chi}^{\dagger} M$ $\mathcal{D}$-mod on $Hecke_\chi$, take

$$p_{1\chi}^{*} M := i_{\chi !*} (p_{1\chi}^{\dagger} M).$$

$$T_\chi M := p_{2x*} \, p_{1\chi}^{*} M$$ - object of derived category...

$$\Rightarrow T_\chi : D(Bun_G) \longrightarrow D(Bun_G \times X)$$

Locally over $Bun_G$, using trivialization $Hecke_\chi$ is product of $Orb_\chi$ by base $\Rightarrow$ $p_{1\chi}^{*} M = M \boxtimes I_\chi$.

Assume we have a ${}^LG$-loc sys $\phi$ on $X$.
We say $M$ on $Bun_G$ is a Hecke $\phi$-eigenmodule if, for $V_\chi$ ${}^LG$-rep of high weight $\chi$ & twist $V_{\chi\phi}$, we have $T_\chi M \simeq M \boxtimes V_{\chi\phi}$.

Comment: everything so far makes sense for $l$-adic sheaves, over fin fields $\Rightarrow$ function which is Hecke eigenfunction with eigenvalues coming from $V$.

The Theorem 1 on Hecke eigenstate property can be stated for all $\phi$ at once: define $\mathcal{D}$-module $V_{\chi A^{gl}}$ on $X$, in fact $A^{glob} \otimes \mathcal{D}_X$-module..
- twist universal oper on $\operatorname{spec} A^{glob}$ by $V_\chi$ get $A^{glob} \otimes \mathcal{D}_X$-module.

$$\Rightarrow T_\chi (\omega_{Bun_G}^{-\frac{1}{2}} \mathcal{D}') = \omega_{Bun_G}^{-\frac{1}{2}} \mathcal{D}'_{Bun_G} \boxtimes_{A^{glob}} V_{\chi A^{glob}}$$

This follows from corresponding local statement:

[ Note that $I_\chi$ $\mathcal{D}$-mod can be characterized as $H^0(i_*(\mathbb{C}_\chi))$ - define $i_{\chi !*} = H^0 i_{\chi *}$
- true due to simply-connected even-dim strata ]

$Y_1 \subset Y_2 \subset Y_3$

Local story: Fix $x$, study $\mathcal{D}$-mods on $GR$.

Formally smooth ind-scheme of infinite type ...

$GR = \bigcup Y_i$ of schemes. We define an

$\mathcal{O}^!$-module - $\mathcal{O}$-module on whole thing any section of which is supported on some $Y_i$:

$M$ is a collection $(M_i)$, $M_i \hookrightarrow M_{i+1}$ ($M_i = i^! M_{i+1}$).

Define $M \otimes \mathcal{D}_{GR} = \mathrm{Diff}(\mathcal{O}_{GR}, M) := \bigcup \mathrm{Diff}(\mathcal{O}_{Y_i}, M_i)$. - exact functor in $M$, by formal smoothness

A $\mathcal{D}$-module structure on $M$: $M \otimes \mathcal{D}_{GR} \to M$ satisfying appropriate properties (as in fin dim case).

- only right $\mathcal{D}$-mods make sense (on level of sheaves).

- note even in fin dimensions extension of $\mathcal{D}$-mod on closed subset $X_1 \hookrightarrow X_2$ is naturally right ..

Another type of $\mathcal{O}$-modules occur as projective limits - these don't form abelian category .. we'll only need line bundles of this type ..

e.g. canonical bundle $\omega_{GR}$:

suppose $\bar{g} = g\, G(\mathcal{O}_x) \in GR$

$\to$ fiber $\omega_{GR\,\bar{g}} := \det[\mathrm{Ad}_g(\mathfrak{g}(\mathcal{O})) : \mathfrak{g}(\mathcal{O})]$

- relative determinant ...

$\mathfrak{g}(K_x)$, $\mathrm{Ad}_g\, \mathfrak{g}(\mathcal{O}_x) \subset$, $\supset \mathfrak{g}(\mathcal{O}_x)$

(can define over families - get line bundle over stack ..).

$\omega_{GR}^{\frac{1}{2}}$ exists (again unique up to trivializing a fiber ..)

- local version of Pfaffian ..

$G(K)$ action on $GR$ lifts to action of central extension on $\omega_{GR}$ - on $\mathfrak{g}(K)$ cocycle is given by Killing form ..

Critical extension $\widetilde{\mathfrak{g}(K)}$ acts on $\omega^{\frac{1}{2}}_{GR}$ so that $\mathbb{1}$ acts as $-1$.

Left $\mathcal{D}$-mods twisted by $\omega^{\frac{1}{2}}$ $\leftrightarrow$ right $\mathcal{D}$-mods twisted by $\omega^{-\frac{1}{2}}$ ...

$M$ (right) $\mathcal{D}$-mod on $G\mathcal{R}$ $\Rightarrow$ $M\omega^{-\frac{1}{2}}$

carries $\widetilde{\mathfrak{g}(K)}$ action which is "critical" (now $\mathbb{1}$ acts by 1)..

- recall $\mathfrak{g}(K)$ acts as vector fields on $G\mathcal{R}$ ...

- so sheaf cohomology groups

$H^i(G\mathcal{R}, M\omega^{-\frac{1}{2}})$ are critical $\widetilde{\mathfrak{g}(K)}$ modules.

<u>Example</u> Take $\delta$-functions at distinguished point, $I_0$ (IC of trivial character) $\to$ no higher cohomologies.

$\Gamma(G\mathcal{R}, I_0\omega^{-\frac{1}{2}}) = Vac'$ : generated by $\delta$ with all derivatives, induced rep from $\mathfrak{g}(O) \to \mathfrak{g}(K)$ with critical level.

<u>Theorem 2</u> $\Gamma(G\mathcal{R}, I_\chi \omega^{-\frac{1}{2}})$ is a direct sum of fin. many copies of $Vac'$, & all higher cohomologies vanish.

Actually these modules carry many additional symmetries...

$Aut\, O \supset Aut^\circ O$ acts on everything : on $G\mathcal{R}$,

$Der\, O$ acts on section of any $\mathcal{D}$-mod $m$. If acts on $\omega^{\frac{1}{2}}_{G\mathcal{R}}$ : it fixes base point, so can lift to act trivially on fibers — no central extension

$\Rightarrow$ $H^i(G\mathcal{R}, M\omega^{-\frac{1}{2}})$ are modules over $Der\, O \ltimes \widetilde{\mathfrak{g}(K)}$.

<u>Theorem 3</u> $\Gamma(G\mathcal{R}, I_\chi \omega^{-\frac{1}{2}}) = Vac' \otimes_{A^{loc}} V_{\chi A^{loc}}$

as $Der\, O \ltimes \widetilde{\mathfrak{g}(K)}$-module.

$A^{loc}$ = functions on local opers $\simeq End(Vac')$ by F.F.

Spec $A^{loc}$ carries canonical ${}^LG$-bundle $\phi$ (flat bundle on formal disc $\leftrightarrow$ ${}^LG$ torsor .. forget flag)

$V_\chi \phi$ is twisted version of $\phi$ ($\phi$-twist of $V$).

— equivariant wrt Aut O-action

$\Rightarrow V_{\chi A^{loc}} = \Gamma(\operatorname{Spec} A^{loc}, V_{\chi\phi})$, free $A^{loc}$ module of finite rank with Aut $\mathcal{O}$-action.
- so Der $\mathcal{O} \ltimes \mathfrak{g}(K)$ acts by projection on Der $\mathcal{O}$.

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$G \supset K$, $\mathcal{H} := D(K\backslash G/K)$ derived category (or equivariant for $G$ wrt $K \sim K$ ...)

$\circledast : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ monoidal structure:

$K\backslash G \times G/K \to (K\backslash G/K) \times (K\backslash G/K)$, $K\backslash G \times G/K \xrightarrow{m} K\backslash G/K$

$K\backslash G \times G/K \downarrow K\backslash G \times_K G/K$

$(K\backslash G/K) \times (K\backslash G/K) \xleftarrow{p} K\backslash G \times_K G/K \xrightarrow{\bar{m}} K\backslash G/K$

$k(g_1, g_2) = (g_1 k^{-1}, k g_2)$

$\mathcal{F}_1 \circledast \mathcal{F}_2 := \bar{m}_* p^t (\mathcal{F}_1 \boxtimes \mathcal{F}_2)$
has associativity constraint etc.

i) $X$ smooth with $G$-action, $D(K\backslash X)$ is an $\mathcal{H}$-module
ii) Harish-Chandra pair $(\mathfrak{g}, K)$
$\Rightarrow$ derived category of $(\mathfrak{g}, K)$-mod $D(\mathfrak{g}, K)$ is a $\mathcal{H}$-module
Simplest example - $K$ trivial, $\mathcal{H}$-modules are
$\mathcal{D}$-mods on $G$, e.g. $\delta$-function at at $g$ $\delta_g$,
$\delta_g(V) = $ "$\mathrm{Ad}_g$" $V$ "turn" of $V$

<u>Reason</u>: Consider $\mathcal{D}$-mods on $G/K$ with $G$-action - not quite equivariant $\mathcal{D}$-mod: acts on module as an $\mathcal{O}$-mod, with compatibility with diff ops...
- weakly equivariant $\mathcal{D}$-modules $M(G \backslash G/K)$

<u>Claim</u> This category is canonically equivalent to $\mathcal{M}(\mathfrak{g}, K)$

- pull back our $\mathcal{D}$-mod to $G$, consider glob sections, which have left $G$-action... $\Rightarrow (\mathfrak{g}, K)$ structure & $\mathcal{H}$ action on $\mathcal{M}(\mathfrak{g}, K)$

(take fiber of $\mathcal{D}$-mod at origin..?)

**Example** Consider Vacuum H-C module - induced from trivial of $K$, corresponds to $\mathcal{D}_{G/K} \in \mathcal{M}(G : G/K)$

Important formula

$M \in \mathcal{H}$ $M \circledast Vac = R\Gamma(G/K, M_{G/K})$

(as plain $\mathcal{O}$-module) - has $(\mathfrak{g}, K)$ action from left translation...

Variant: $\lambda : K \to \mathbb{G}_m$ character $\Rightarrow Vac_\lambda$.

$M \circledast Vac_\lambda = R\Gamma(G/K, M_{G/K}(\lambda))$

**Fact** $D(\mathfrak{g}, K) \underset{L\Delta}{\overset{R\Gamma}{\rightleftarrows}} D(K\backslash X)$ , $R\Gamma(M) := R\Gamma(X, M_X)$

$L\Delta(V) = \mathcal{D}_X \overset{L}{\underset{U\mathfrak{g}}{\otimes}} V$ (K-equivariant $\mathcal{D}$-mod)

Then $R\Gamma, L\Delta$ are morphisms of $\mathcal{H}$-modules

**Our situation** $K = K_x$ , $G(K)$ is group ind-scheme, points $G(K)(A) = G(K \hat{\otimes} A)$ . $\mathbb{C}((t)) \hat{\otimes} A = A((t))$

Group subscheme (affine group scheme) $G(\mathcal{O})$

$G(K)/G(\mathcal{O})$ is an ind-scheme of inf-finite type

$\to$ get Hecke algebras etc.

$\mathcal{H} = D(G(\mathcal{O}) \backslash G(K)/G(\mathcal{O}))$

Fix integral level $\lambda \Rightarrow$ Kac-Moody group $G(K)^{\sim} \supset G(\mathcal{O}) \times \mathbb{G}_m$.

$\to$ gives same Hecke algebra's abelian core (whole category will be slightly different)

$\mathcal{H}' \supset \mathcal{M}(G(\mathcal{O}) \backslash G(K)/G(\mathcal{O})) \subset \mathcal{H}$

Even larger Harish-Chandra pair: $\mathrm{Aut}\,\mathcal{O} \ltimes G(K)^{\sim}$, $\mathrm{Aut}^{\circ}(\mathcal{O}) \ltimes G(\mathcal{O}) \times \mathbb{G}_m$

Still same core - equivariant $\mathcal{D}$-mods on double classes.

**Thm** $\circledast$ is t-exact (in any of these settings)

We'll be interested only in action of core $\to$ same algebra.

**Th** $(M(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}), \circledast) \xrightarrow{\sim} (\mathrm{Rep}\,{}^LG, \otimes)$

(in particular → commutative ..)

Fiber functor: $\bigoplus H^i_{DR}(G(\mathcal{K})/G(\mathcal{O}), M)$

Grading on $H_{DR}$ ⟺ grading on principal $sl_2$ of ${}^LG$,

raising operator is multiplication by first chern class of ample line bundle..

Consider $\mathcal{Y} = \widehat{\mathrm{Bun}_G} \times \hat{X} = \{(\mathcal{F}, x, \gamma, t)\}$ + parameter
$\mathrm{Bun}_G$, $\hat{X}$, trivialization $\in \mathcal{F}(\mathcal{O}_x)$

This is a scheme, with action of $G(\mathcal{K})$ & Aut $\mathcal{O}$

Over this have line bundle of half-forms on $\mathrm{Bun}_G$

⇒ $\mathbb{G}_m$-torsor $\mathcal{Y}^\sim \to \mathcal{Y}$

Has action of our large H-C pair.

Quotient of $\mathcal{Y}^\sim$ by $\mathrm{Aut}^\circ\mathcal{O} \ltimes G(\mathcal{O}) \times \mathbb{G}_m$

will be $\mathrm{Bun}_G \times X$.

$L\Delta(Vac')$ is $\mathcal{D}$-mod of critical diffops, untwisted:

$\omega^{-\frac{1}{2}} \mathcal{D}'_{\mathrm{Bun}_G} \boxtimes \mathcal{O}_X$ (Recall $L\Delta: D(\mathfrak{g}, K) \to D(K\backslash X)$ )
$\mathrm{Bun}_G \times X$ for us

Hecke operator $T_x$ ⟷ D-mod supported on $X$ a pt..

Recall local result, ⇒ the representation in Th 3

coincides with $I_X(Vac')$ : follows from

variant of important formula..

Local story Assume Th 2 from last time ..

$\mathfrak{z} = \mathrm{End}\,Vac'$. Consider the functor $\mathrm{Rep}\,{}^LG \to$ free $\mathfrak{z}$-modules,

$V \mapsto I_V$ D-mod on affine Grassmannian $(\in \mathcal{H})$,

apply to Vacuum rep ⇒ $I_V(Vac')$,

$\mapsto \mathrm{Hom}_{\mathfrak{g}(\mathcal{K})}(Vac', I_V(Vac'))$

→ free finite rank $\mathfrak{z}$-module, $V_\phi$.

Claim This is a fiber functor on $\mathrm{Rep}\,{}^LG$ (tensor functor.)

Note $\boxed{I_V(Vac') = V_\phi \otimes_{\mathfrak{z}} Vac'}$

→ get compatibility with tensor product. compatibility with commutativity less obvious.

- This is same as giving ${}^LG$ - bundle on Spec $\mathfrak{Z}$, denoted $\phi$. This is Aut $\mathcal{O}$ -equivariant (by naturality).

<u>Extra structure</u>: Can reduce $\phi$ canonically to ${}^LB \subset {}^LG$, not compatible with Aut $\mathcal{O}$ (but with $\mathrm{Aut}^\circ \mathcal{O}$).

- What does it mean to give a reduction to Borel?
  - V $\phi$rep contains unique high weight line wrt B
- reduction determined by line in each $\phi$twist of an irrep.

⇒ Need canonical line. <u>Basic Fact</u>: $Orb_\chi \subset GR$, restrict $\omega_{GR}^{\frac{1}{2}}$ to $Orb_\chi$ is ≃ the <u>canonical</u> (not $(\ )^{\frac{1}{2}}$) bundle of $Orb_\chi$ : $\bar{g} \in Orb_\chi$

$$\omega_{GR}^{\frac{1}{2}}(\bar{g}) = \det^{-1}\left[ {}^{\mathfrak{g}(\mathcal{O})}\big/_{Ad_g(\mathfrak{g}(\mathcal{O})) \cap \mathfrak{g}(\mathcal{O})} \right] = \omega_{Orb_\chi}(\bar{g})$$

$Orb_\chi \overset{j}{\subset} \overline{Orb_\chi}$, $j_\cdot \omega_{O_\chi} = I_\chi$

Untwist ⇒ $j_\cdot \mathcal{O}_{O_\chi} = I_\chi \, \omega_{GR}^{-\frac{1}{2}}$ - produces global section of $I_\chi$, hence a line

- gives generator of 1-dim $\mathfrak{Z}$-module

⇒ reduction to Borel. Preserved by $\mathrm{Aut}^\circ \mathcal{O}$ — that preserves our stratification .. but all Aut $\mathcal{O}$ doesn't preserve it ...

Consider Aut $\mathcal{O}$ orbit of $z \in$ Spec $\mathfrak{Z}$, restrict $\phi \supset \phi^{(B)}$ to this orbit ... ⇒ B-bundle on

$\mathrm{Aut}^\circ \mathcal{O} \backslash \mathrm{Aut}\, \mathcal{O} z$ = the formal disc!

Corresponding ${}^LG$ bundle has a trivialization (Aut $\mathcal{O}$ equivariant) - flat connection...

- this is an oper!

⇒ map Spec $\mathfrak{Z}$ ⟶ Spec $A^{loc}$

<u>Theorem</u> This is exactly the F-F isomorphism.

## Local story II "Renormalized" enveloping algebra.

Denote $U = U g(K)$. $\mathcal{D} = \Gamma(GR, \mathcal{D}')$

(something that acts on any $\mathcal{D}$-module).

We have map $U \to \mathcal{D}$., neither $\hookrightarrow$ nor $\twoheadrightarrow$.

$U$ has center $\mathfrak{Z}^{x}$ $\supset$ ideal $I$ which kills Vac:

$I$ maps to zero in $\mathcal{D}$ - diffop that acts trivially on $\delta$-functions at base, but is $g(K)$-equivariant (it's in center) so it acts trivially on $\delta$-functions at any point.

In finite dimension $U \to \mathcal{D}$ is onto on assoc graded (study moment maps $\to \mathcal{N}$ nilcone).

It's not onto - have e.g. action of change of coords - noncritical level have Sugawara operators, but not at crit...

$\Rightarrow$ Enlarge $U/I \longrightarrow \mathcal{D}$ to make all our modules irreducible...

$\searrow U^{\#} \nearrow$

- Change level, $U_\lambda$ = Univ env. at level $\lambda$ + crit.

$U_\lambda \longrightarrow \mathcal{D}_\lambda$ flat wrt $\lambda$.

Consider $\lambda^{-1}\tilde{I}$ : $\tilde{I}$ is part whose reduction mod $\lambda$ is in $I$. $\tilde{I} \to \lambda \mathcal{D}_\lambda$, get compatible map $\lambda^{-1}\tilde{I} \longrightarrow \mathcal{D}_\lambda$.

- lies in localization of $U_\lambda$ by $\lambda$.

$U_\lambda^{\#}$ is subalgebra of localization generated by $\lambda^{-1}\tilde{I} + U_\lambda$

$P$ is image of $\lambda^{-1}\tilde{I} + U_\lambda$ in $U^{\#}$.

- it is a Lie subalgebra, normalizing $U/I$.

$$0 \to U/I \to P \to I/I^2 \to 0$$

$\Rightarrow$ map $I/I^2$ to act on $\mathfrak{Z}$ in $U/I$...

$I/I^2$ is a $\mathfrak{Z}$-module $\Rightarrow$ in fact it is a Lie algebroid acting on $\mathfrak{Z}$.

<u>Claim</u> Wrt the F-F identification $\mathfrak{Z} \xrightarrow{\sim} A^L$

$I/I^2 \xrightarrow{\sim}$ Lie algebroid of inf automorphisms of the tautological $\xi$ bundle

in particular acts transitively on Spec $\mathfrak{Z}$, + twisted form of $^L g$.

$\Rightarrow$ Irreducibility of Vacuum follows immediately;
Same for irred V.. have transitive Lie algebroids on Spec $\mathfrak{Z}$, Spec $A^{\infty}$ & compatibility — almost forces our map to be FF isomorphism...

Can see in $I/I^2$ part that preserves $\beta$-structure.

Heisenberg dual: adjoint action of $I/I$ on $\mathfrak{Z}$ realizes action of derivatives..