A. Beilinson   Differential Operators on Ind-Schemes

~~B~~ k fixed base field. topological vector spaces: separated & complete.
3 types of $\otimes$ on these:

V_i finite collection of top. vect spaces
$\otimes V_i$ carries several topologies $\Rightarrow$ different completions

$\otimes^* V_i$: strongest possible natural topology $\{\otimes^* V_i \to W\}$ = continuous polylinear maps
$\otimes^! V_i$ weakest : ~~B~~ limits of finite things (discrete)

Intermediate ones depend on total ordering of indices:

$V_1, \ldots V_n \Rightarrow V_1 \vec{\otimes} V_2 \vec{\otimes} \ldots \vec{\otimes} V_n$
defined by induction : completion wrt topology on $\otimes V_i$. Define topology by giving morphisms to discrete v.s. F

allow $V_1 \otimes \ldots \otimes V_n \xrightarrow{\phi} F$ discrete s.t.

- $\exists\, U \subset V_n$ open, s.t. $\phi(V_1 \otimes \ldots \otimes V_{n-1} \otimes U) = 0$
- for $\forall v \in V_n$, the polylinear maps
$$V_1 \otimes \ldots \otimes V_{n-1} \xrightarrow{\phi_v} F \text{ is}$$
continuous wrt topology defined by induction

Some of these topologies $(\otimes^*, \vec{\otimes})$ are not countable – hard to describe via projective limits.

~~These $\otimes$'s differ~~

Dennis

If $V_2 = \text{``}\varprojlim\text{''} W^i$ discrete (pro-vector spaces)

$\Rightarrow V_1 \vec{\otimes} V_2 = \varprojlim_i (V_1 \vec{\otimes} W^i)$ projective limit in category of pro-vector spaces

$\mathbb{V}$ pro-vector space, W vector space

$\Rightarrow \mathbb{V} \otimes W = \varinjlim_{\substack{W_0 \subset W \\ \text{fin dim}}} (\mathbb{V} \otimes W_0)$ inductive limit of pro-vector spaces — ie <u>noncompleted</u> tensor product

Vect = Ind (fin-dim vect).

taking ind in pro-vect is not nice... just as taking pro-limits in vector spaces is not nice...

Sasha

Natural maps $\otimes^* \longrightarrow \vec{\otimes} \longrightarrow \otimes^!$

...more convenient to take

$$\underset{I}{\otimes^*} \longrightarrow \bigoplus_{\substack{\text{all linear} \\ \text{orderings of } I}} \underset{I}{\vec{\otimes}} = \otimes^{ch}$$

Polylinear operations on $\otimes^{ch}$ naturally compose.

<u>Geometric origin of $\otimes^{ch}$</u> : Chiral operations

X a curve, $x \xhookrightarrow{i_x} X \xhookleftarrow{j_x} U_x$ complement

M $\mathcal{D}_X$-module

Consider all possible quotients of M supported at x

$\longleftrightarrow$ all $P \subset M$: $P|_{U_x} = M|_{U_x}$

– form a filter ($\Rightarrow$ topology) on M: intersection of such is again such, anything containing such is as well.

$h(M/p)$ vector space ($\mathcal{D}$-mod supported at $x \xrightarrow{h} Vect$)
(deRham cohomology) (or deRham cohomology on open disc at x)

Take $\hat{h}_x(M) = \varprojlim_{p} h(M/p)$: $h(P)$ form a topology on $h(M)$
& this is the completion of $h(M)$ wrt this topology

$\{M_i\}_{i\in I}$ collection of $\mathcal{D}$-modules, N another $\mathcal{D}$-module.

Consider $X \xrightarrow{\Delta^{(I)}} X^I \xhookleftarrow{j^{(I)}} U^{(I)}$

$$\mathrm{Hom}\left(j_*^{(I)} j^{(I)*}(\boxtimes M_i) ; \Delta_*^{(I)} N\right) = P_I^{ch}(\{M_i\}, N) \text{ chiral operations}$$

* Claim $j_*^{(I)} j^{(I)*}(\boxtimes j_{x*} M_i)$ has a natural quotient
[M_i are $\mathcal{D}$-mod on U_x, extend as $j_{x*}M_i$] equal to
$(\Delta^{(I)}_x)_* \otimes^{ch} \hat{h}_x(j_{x*} M_i)$ (sitting at $(x, x, \ldots, x)$)

.. this is not the maximal quotient supported at $(x, \ldots x)$, just some quotient.
[Note: weaker topologies will define further quotients]

have Weaker topologies instead of $\hat{h}_x(M)$, can say when a given chiral operation is continuous wrt these topologies: ask for the operations to factor through continuous maps on this quotient space.

A priori: chiral operations need not be continuous wrt any of the standard topologies — almost never continuous wrt natural topologies. but for chiral algebra have continuity wrt topology formed by chiral subalgebras.

Some morphisms ... "continuous" ... descend to give morphisms on $\mathfrak{h}_x$:

$$\begin{array}{ccc} j_x^{(I)} j^{(I)*} \boxtimes j_{x*} M_i & \longrightarrow & \Delta_x^{(I)} j_{x*} N) \\ \downarrow & & \downarrow \\ \otimes^{ch} \mathfrak{h}_x^{\wedge} & \longrightarrow & \mathfrak{h}_x^{\wedge} N \end{array}$$

Given a chiral algebra, have topology formed by <u>chiral subalgebras</u>, claim: chiral operation is continuous <u>wrt this topology</u>

<u>Pf of Claim *</u> Fix $P_i \subset j_{x*} M_i$, coinciding outside of x, $i_{x*} T_i := j_{x*} M_i / P_i$ corresponding vector space

$$\Rightarrow \quad 0 \to \boxtimes P_i \big|_{U^{(I)}} \longrightarrow \boxtimes j_{x*} M_i \big|_{U^{(I)}} \longrightarrow$$

$$\twoheadrightarrow \bigoplus_{i \in I} \Big(\boxtimes_{i' \in I \setminus \{i\}} j_{x*} M_{i'} \Big) \boxtimes i_{x*} T_i \to 0$$

Now extend with poles to X^I : get a canonical quotient

$$j_x^{(I)} \big(\boxtimes j_{x*} M_i \big) \big|_{U^{(I)}} = \bigoplus j_x^{(I \setminus i)} \Big(\boxtimes j_{x*} M_{i'} \big|_{U^{I \setminus i}} \Big) \boxtimes i_{x*} T_i$$

Now use induction hypothesis -- claim on $I \setminus i$ ... & take projective limit over all P_i to get our desired quotient.

$\blacksquare$

X reasonable formally smooth ind-scheme

A cdo $\mathcal{D}$ on X $\longmapsto$ a chiral extension
$$0 \to \mathcal{O}_X \to \mathcal{D}_1 \to \Theta_X \to 0$$
$$\tau \mapsto \bar{\tau}$$

extension of topological vector spaces, + topological $\mathcal{O}_X$ bimodule -- using $\vec{\otimes}$ tensors --

+ Lie* algebra structure, wrt $\otimes^*$

Denote $\mathcal{D}_1 = \Theta_X^\flat$

$\tau \in \Theta_X^\flat$, $f \in \mathcal{O}_X \Rightarrow \tau f - f\tau = \bar{\tau}(f) \in \mathcal{O}_X \subset \mathcal{O}_X^\flat$

$[\tau, f\tau'] = \tau(f)\tau' + f[\tau,\tau']$, same for $f\tau' \rightsquigarrow \tau' f$

& $1 \in \mathcal{O}_X \subset \Theta_X^\flat$ central.

(Cdos) $\longrightarrow$ (chiral extensions) is fully faithful
... presumably an equivalence ("PBW")

Chiral extensions are a torsor over the Picard groupoid of
Naive extensions: $0 \to \mathcal{O}_X \to \Theta_X^{\text{naive}} \to \Theta_X \to 0$

- Θ_X^{naive} Lie* algebra extension
- Θ_X^{naive} $\mathcal{O}_X$-module ~~[illegible]~~ ... action continuous wrt $\otimes^!$ action ... & a Lie* algebra

ie replace $\vec{\otimes}$ bimodule by $\otimes^!$ module ...

Claim Naive extensions are Picard groupoid ... have Baer sum etc

Baer sum of chiral + naive is chiral
Baer difference of chirals is naive ....

So the groupoid of chiral extensions (if nonempty) is a Torsor wrt Picard groupoid of naive extensions.

Why? take Baer difference of chiral extensions
$\Rightarrow$ $\mathcal{O}_X$-bimodule, but now symmetric:
obstruction to symmetricity $\tau f - f\tau = \overline{\tau}(f)$ is same for both, so cancels in difference — central bimodule
--- continuous wrt $\vec{\otimes}$ in both orders
$\Rightarrow$ continuous wrt $\otimes$!

How to define cdos? on group we used left, right translations by Lie algebra.

Generally: $\mathfrak{g}$ a Tate Lie*algebra (Lie algebra, continuous in * sense) & acts on space X
$\Rightarrow$ morphism $\mathfrak{g} \longrightarrow \Theta_X$

Extend to $\mathcal{O}_X \otimes^! \mathfrak{g} \longrightarrow \Theta_X$, formally simply transitive if this is an isomorphism

Lemma Suppose $\mathfrak{g}$ action on X is formally simply transitive
$\Rightarrow \exists!$ chiral extension $0 \to \mathcal{O}_X \to \Theta_X^b \to \Theta_X \to 0$

$$\begin{tikzcd} \Theta_X^b \arrow[r] & \Theta_X \\ & \mathfrak{g} \arrow[ul, dotted] \arrow[u] \end{tikzcd}$$

equipped with a lifting of $\mathfrak{g}$ compatible with brackets

How to construct?
Write
$$0 \longrightarrow \mathcal{O}_X \otimes^* \mathfrak{g} \longrightarrow \mathcal{O}_X \otimes^{ch} \mathfrak{g} \longrightarrow \mathcal{O}_X \otimes^! \mathfrak{g} \longrightarrow 0$$
$$\mathcal{O}_X \otimes^{ch} \mathfrak{g} = \mathcal{O}_X \vec{\otimes} \mathfrak{g} \oplus \mathcal{O}_X \overleftarrow{\otimes} \mathfrak{g}$$

(Have such exact sequence for any two topological vector space)

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \otimes^* \mathfrak{g} \arrow[r] \arrow[d, "\text{action of } \mathfrak{g} \text{ on } \mathcal{O}"] & \mathcal{O}_X \otimes^{ch} \mathfrak{g} \arrow[r] \arrow[d, "\text{pushout}"] & \mathcal{O}_X \otimes^! \mathfrak{g} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & \Theta_X^b \arrow[r] & \Theta_X \arrow[r] & 0
\end{tikzcd}
$$

Conversely $\mathfrak{g}$ sits in Θ_X^b $\Rightarrow$ can multiply from left & from right by functions, recover upper short exact.

This gives Θ_X^b as topological vector space.

Note that

$$
\begin{tikzcd}
 & \Theta_X & \\
\mathcal{O}_X \overleftarrow{\otimes} \mathfrak{g} \arrow[ur, hook] & & \mathfrak{g} \overrightarrow{\otimes} \mathcal{O}_X \arrow[ul, hook']
\end{tikzcd}
$$

Lie* subalgebras (always not closed)

In fact ~~$\mathfrak{g} \overrightarrow{\otimes}$~~ they are ~~normal~~ Lie* subalgebras (~~...~~).
& $\mathfrak{g} \overrightarrow{\otimes} \mathcal{O}_X$ is normal (ideal)

Concretely:

$$\mathcal{O}_X \overrightarrow{\otimes} \mathfrak{g} : \Big\{ \sum_{\tau_i \to 0} f_i \otimes \tau_i \Big\}$$

$$\mathcal{O}_X \overleftarrow{\otimes} \mathfrak{g} : \Big\{ \sum_{g_i \to 0} g_i \otimes \eta_i \Big\}$$

Any vector field is sum of two such: these span, & we know their relations

We have sections

$$
\begin{tikzcd}
\Theta_X^b \arrow[r] & \Theta_X \\
\mathcal{O}_X \overleftarrow{\otimes} \mathfrak{g} \arrow[u] \arrow[ur, hook] & \mathfrak{g} \overrightarrow{\otimes} \mathcal{O}_X \arrow[u, hook'] \arrow[ul, bend left]
\end{tikzcd}
$$

Compatible with bracket
in fact ~~they~~ define bracket on Θ_X^b.

Now define bimodule structure ...

$$\Big[\sum_{\tau_i \to 0} f_i \otimes \tau_i \,,\ \sum_{g_j \to 0} g_j \otimes \eta_j \Big] \in \mathcal{O}_X \overleftarrow{\otimes} \mathfrak{g}$$

$f\tau(g) \otimes \eta$: $f\tau(g) \to 0$ if $\tau, g \to 0$.

X ind scheme, ~~reasonable~~ formally smooth, $\Omega = \mathrm{Sym}^!(\Omega^1_X[-1])$ ∞-dim exterior algebra

[Need to trivialize dlog of determinant gerbe of cotangent bundle to define cdo . . . .]

Ω is a graded superalgebra
Spec $\Omega^\cdot$ graded super ind scheme.
$= \Pi T_X$. As dg scheme, with de Rham differential : denote it by $DR(X)$

Claim $\exists!$ DG cdo $\mathcal{D}_{DR}$ / $DR(X)$

--- $\mathbb{Z}$-graded, with odd differential

(unique, exists modulo PBW : we'll only construct the corresponding chiral extension).

Extra rigidification : d itself is element of $\mathcal{D}_{DR}$.

If X is a Tate scheme (each closed subscheme locally product a scheme of finite type by ∞-dim affine space) then no problem.

$\Rightarrow$ Formal smoothness : condition only on $0, 1^{st}$ pieces of cotangent complex. In fin dim this implies all cohomologies of cotangent complex vanish, i.e smoothness

Why? $\Omega X \downarrow X$ $\Rightarrow$ short exact sequence on tangents

$$0 \to \Theta_{\Omega X/X} \to \Theta_{\Omega X} \to \Omega \otimes^!_{\mathcal{O}_X} \Theta_X \to 0$$

~~gradr~~ $\Theta_{\Omega X/X}$ generated ~~b~~ freely by elements in degree -1:

$$\Theta_{\Omega X/X} = \Omega_X \otimes^!_{\mathcal{O}_X} \Theta_X [+1]$$

So lowest component of $\Theta_{\Omega X}$ is $\Theta_X[1]$.

Differential gives isomorphism of subbundle to quotient bundle ("oper condition" --- strict Griffiths transversality for differential)).

$$\Theta_X[1] \xrightarrow[\sim]{d} \Theta_X[0] = \cancel{\Theta^0_{\Omega X}}$$

deg zero part $\Theta^0_{\Omega X} \simeq \Theta^0_X \oplus (\Omega^1 \otimes^! \Theta_X)$

0 component of the chiral extension: piece of Clifford algebra.

Components of degree -1 & 0 are bimodule over $\mathcal{O}_X$...

Claim makes sense in finite dimensions: $\exists!$ DG cdo...
say some words in ∞ situation (with some complications).
Main point: construction makes sense, since left & right module structures are continuous.
... same as construction of $\mathcal{D}_{\leq 1}(\Omega^\bullet)$
... Clifford algebra of contracting multiplication & Lie action by vector fields.

Need de Rham differential! Lie action is arising by image $\Theta_X^{-1} \xrightarrow[d]{\sim} \Theta_X^0$.

$\mathcal{D}_X$-mod := modules over this algebra
Need PBW to know there are nonzero $\mathcal{D}$-modules ...

ΠT_X: add odd coordinates — nilpotent! so category of $\mathcal{D}$-modules is same as on X itself... so this gives way to define $\mathcal{D}$-modules in ∞ dim, even when there are no cdos on X itself.

This takes a left $\mathcal{D}$-mod to its deRham complex
That's what we get as primary objects in ∞ dim...
Might not admit a grading in ∞ dimensions
– since it's not coming from $\mathcal{O}_X$-module: don't have beginning or end of the deRham complex

To get $\mathcal{O}_X$-module of shape we want: need to pick a fermion module – discrete module for $Cliff(\Theta \oplus \Omega)$,
– want its restriction to any point to be irreducible.
This fermion module may not exist!
if it exists, unique up to twisting by a line bundle
– so fermion modules form torsor over Picard category of line bundles.

A choice of such defines cdo on X & equivalence between modules over this cdo & $\mathcal{D}_{DR(X)}$ ....
Finite dim: dR complex $\Longrightarrow$ Hom_{Cliff}(fermion module, dR M) is TDO module on X....

$$0 \to Cliff_1^0 \to \Theta_{DR}^{b,0} \longrightarrow \Theta_X \to 0$$

degree zero

[where $0 \to \mathcal{O}_X \to Cliff_1^0 \to \Omega \otimes \Theta \to 0$ order 1 piece]

Let V be a fermion module.

Consider $\tilde{\Theta}_X = \{(\tau \in \Theta_X, \tilde{\tau}$ its lift to an action on $V)\}$ as $\mathcal{O}$-module

τ acts on Cliff, demand $\tilde{\tau}$ commutes with Cliff in some fashion

Now look at pairs of elements of Θ_{DR}^{b0} & of $\tilde{\Theta}_X$ with same underlying vector field & same commutation relation with the Clifford algebra

Such pairs form an extension of Θ_X by $Cliff_1^0$

M $\mathcal{D}_{DR}$-module $\Rightarrow$ $Hom_{Cliff}(V, M)$ carries action of cdo on X.

How to go from V to CDO on X?

First consider larger extension, consisting of triples $(\tau, \tilde{\tau}_{DR}, \tilde{\tau}_V)$: $\tau \in \Theta_X$, $\tilde{\tau}_{DR} \in \Theta_{DR}^{b0}$, $\tilde{\tau}_V$ is an action of τ on V

s.t. the derivation of $Cliff_1^0$ given by adjoint action $Ad\,\tilde{\tau}_{DR}$ determines the commutation between $\tilde{\tau}_V$ & $Cliff_1^0$ acting on V

$$[\tilde{\tau}_V, \gamma] = Ad\,\tilde{\tau}_{DR}(\gamma) \qquad \gamma \in Cliff.$$

Cliff sits inside these triples, inside the locus $\tau = 0$ ... in fact it is normal.

Take quotient $(\tau, \tilde{\tau}, \tilde{\tau}_v) / \langle Cliff \rangle$ =: our chiral extension on X.

— Note all this story same in finite dimensions!

All choices of families of irred Clifford modules on X differ by twist by line bdles..