Geometric Langlands Seminar

A. Beilinson – Introduction 10/7/99

Plan:
- Geometric Eisenstein Series (Braverman–Gaitsgory)
- Casselman–Shalika formula (Frenkel–Gaitsgory–Kazhdan–Vilonen)
- Semi-infinite Grassmannian
- Bernstein Center
- Higher dimensions (Kapranov)

Automorphic Forms F global field $\begin{cases} \text{e.g. number field: finite ext of } \mathbb{Q} \\ k(X),\ k \text{ finite } X/k \text{ curve} \end{cases}$

$F \subset \mathbb{A}$ = adèles of F G/F reductive (e.g. GL_n)
(discrete) $\cup$
$\mathcal{O} = \mathcal{O}_{\mathbb{A}} = \prod \mathcal{O}_v$ $G(\mathbb{A})$ locally compact $\supset G(F)$ discrete
$\to$ invariant measure

G semisimple e.g. $\Rightarrow \mathrm{vol}\,(G(\mathbb{A})/G(F)) < \infty$

$G(\mathbb{A}) \circlearrowright L^2(G(\mathbb{A})/G(F))$ – study module structure. [unitary]

e.g. $G = GL_2$ modular functions sit in here...

Totally disconnected $\to$ consider functions which are locally constant
$\rightsquigarrow$ invariant under some open subgroups –
defined by congruence conditions in $G(\mathcal{O}_{\mathbb{A}})$
– e.g. $D \subset X$ effective divisor
$\Rightarrow \mathcal{O}_{\mathbb{A}D} \subset \mathcal{O}_{\mathbb{A}}$ integral adèles vanishing along D + constants

$G(\mathcal{O}_{\mathbb{A}}) \supset G(\mathcal{O}_{\mathbb{A}D})$ [corresponds to max ideal in $\mathcal{O}_{\mathbb{A}D}$ – elements $1 + G(\mathcal{O}_{\mathbb{A}D})$: congruent to 1 mod $\mathcal{O}_{\mathbb{A}D}$]

$\to$ basis of open nbhds of $\mathcal{O}$.

L^2 is unitary $\to$ (cont) sum of irreducibles, which are tensor of irreds for local groups
$\mathbb{A} \subset \prod K_v$

$$M = \bigotimes M_v \quad \text{local irreds}$$

– any vector fixed by local G's for almost all places.
M_v contains almost unique $G(\mathcal{O}_v)$-fixed vectors.

Langlands – explain restrictions on M_v which come this way from adèlic representation.

Local Story

First consider reps with $G(\mathcal{O}_v)$ fixed vector.
$G(\mathcal{O}_v) \subset G(K_v)$, M $G(K_v)$-module (continuous wrt discrete topology)

$M^{G(\mathcal{O}_v)}$ would be rep of quotient gp if $G(\mathcal{O}_v)$ were normal,
– but does carry Hecke algebra action
$\mathcal{H}(G(K_v), G(\mathcal{O}_v))$

every vector fixed by open subgroup $\longleftrightarrow$ considering locally constant functions

Action of $G(K_v)$ on M $\longleftrightarrow$
action of algebra of all measures, compact support, on $G(K_v)$ wrt convolution, which are invariant wrt some open subgroup. [algebra without unit]

$\mu \in \mathcal{H}$ $m \in M$ $\mu(m) = \int_G g \cdot m \, d\mu$ ← loc invariant
← loc constant M-valued fn.

e.g. $g \in G$ ~ δ-fn at g gives g action on M.

[• finite group G reps $\longleftrightarrow$ rep of $\mathbb{C}[G]$
– this is analogue.]

$\mathcal{H}(G(K), G(\mathcal{O})) \subset \mathcal{H}$: $G(\mathcal{O})$-bi-invariant measures
(nonunital) subalgebra of $\mathcal{H}$.
– preserves $M^{G(\mathcal{O}_v)}$

$$\left\{ \begin{array}{l} \text{modules gen. by } G(\mathcal{O})\text{-invariant} \\ \text{vector} \end{array} \right\} \longrightarrow \mathcal{H}(G(K), G(\mathcal{O}))\text{-mod}$$

$$M \longmapsto M^{G(\mathcal{O}_v)}$$

sometimes equivalence of categories.

<u>Satake</u> Assume G split (e.g. GL_n) (e.g. anything over alg closed field)

T Cartan torus (canonical) assigned to G
$N \subset B \subset G \longrightarrow T = B/N$, indep of choice of B
$(\mathrm{Norm}(B) = B \ldots)$

- any max torus conjugate to any other, but in many different ways — fix B to fix W action.

Γ corresp lattice $\quad T = \Gamma \otimes \mathbb{G}_m$, as groups.

To get Γ from T: $\Gamma = \mathrm{Hom}(\mathbb{G}_m, T)$

Local fields: can reconstruct Γ as $T(K_v)/T(\mathcal{O}_v)$.
$(K_v^* / \mathcal{O}_v^* = \mathbb{Z})$

dual (~) Tori $\leftrightarrow$ lattices $\circlearrowleft$ dual $= \mathrm{Hom}(-, \mathbb{Z})$

$$T^\vee = \mathrm{Spec}\, \mathbb{Q}[\Gamma] \quad . \quad \Gamma^\vee = \mathrm{Hom}(T, \mathbb{G}_m)$$

$$T^\vee(A) = \mathrm{Hom}(\Gamma, A^*) \subset \mathrm{Hom}(T(K_v), A^*)$$

so $T^\vee(A) \longleftrightarrow$ unramified characters of $T(K_v)$
(factor through $T(K_v)/T(\mathcal{O}_v)$)

<u>Satake isomorphism</u> $\mathcal{H}(G(K_v), G(\mathcal{O}_v)) \xrightarrow{\sim} \mathcal{O}(T^\vee_{\mathbb{C}})^W$
(complex-valued) $\qquad$ regular alg. fns.

<u>in particular</u> $\mathcal{H}(G(K_v), G(\mathcal{O}_v))$ is commutative
($\Rightarrow$ irred modules 1-dim, so if $M^{G(\mathcal{O}_v)} \neq 0 \Rightarrow$ 1-dimensional!)

T carries extra structure from G — root system.
$\rightsquigarrow$ dual root system for $T^\vee \longrightarrow G^\vee$ Langlands dual.
- over $\mathbb{Q}$, $\quad T^\vee \subset G^\vee$ canonically.

$\mathcal{O}(T^\vee_{\mathbb{C}})^W$ W-inv fns on $T^\vee \xleftrightarrow{\sim} G^\vee$-inv fns on $G^\vee$. $\mathcal{O}(G^\vee_{\mathbb{C}})^{Ad}$

<u>Satake map</u>: to any element of torus $T^\vee$
define irrep of $G(K_v) \rightarrow$ eigenvalue of $\mathcal{H}(\,,\,)$
which is value of corresponding function on $T^\vee$
at that element.

X alg. variety over F_v. $\mathcal{C}(X(F_v))$ vect. space
of locally constant functions (topology comes from that of F_v).

$G \circlearrowright X \Rightarrow$ action (cont.) on $\mathcal{C}(X(F_v))$.

<u>Twisted version</u> $\quad \chi : F_v^* \longrightarrow \mathbb{C}^*$

$\varphi \in \mathcal{O}^*(X)$ invertible $\Rightarrow$ $\varphi_\chi = \chi(\varphi|_{X(F_v)})$
locally constant function on $X(F_v)$, values in $\mathbb{C}^*$.

Then $\mathcal{L}$ line bundle $/X$ $\longmapsto \mathcal{L}_\chi$ $\mathbb{C}$-line bundle on set of F_v points (apply χ to transition functions.

multiplicative in $\mathcal{L}$

$X(F_v)$ completely disconnected - so line bundle is trivial but not canonically - & will get some interesting structure

$\Rightarrow$ for $\mathcal{L}$ equivariant, $\mathcal{C}(X(F_v), \mathcal{L}_\chi)$ is G-representation.
(Similarly for torus characters & T-bundles)

<u>Example</u> i)
ω_X bundle of forms, G-equivariant ($G \curvearrowright X$)
$\chi = |\,|$ modulus $\chi(x) = |x|$.

$\Rightarrow$ $\omega_{X,|\,|}$ has as sections locally constant measures on $X(F_v)$.
e.g. for X smooth local coords give standard measure, & they transform as above.
(archimedean case e.g.)

ii) Pick $\omega_X^{\frac{1}{2}}$ $\omega^{\frac{1}{2}}_{X,|\,|}$ half-measures - form a pre-Hilbert space. (real case this is canonical $+\sqrt{|x|}$)

$X =$ flag variety G/B if fix B.

$\mathcal{L} = \tilde{X} = G/N \circlearrowleft_B$ $N = \mathrm{rad}(B)$ "marked flags"
$\tilde{X} \nearrow$ T-torsor; B from right

Any character of $T(k)$ produces $G(F_v)$-equiv line bundle on G/B, whose space of sections is the induced rep

$\chi: T(K_v) \to \mathbb{C}^*$ $\Rightarrow$ $G(K_v)$-equiv line bundle $\mathcal{L}_\chi / X(F_v)$
$\Rightarrow$ $\mathcal{C}(X(F_v), \mathcal{L}_\chi)$ $G(K_v)$-module

(twisted by $\omega^{\frac{1}{2}}$ $\Rightarrow$ unitarily induced rep $\mathrm{Ind}(\chi)$)

Reason for twist: if $\chi : F_v^* \to \mathbb{C}^*$ unitary $\Rightarrow \mathcal{L}_\chi$ has canonical measure $\to$ $C(X(F_v), \mathcal{L}_\chi \cdot \omega_X^{\frac{1}{2}})$ unitary rep.

$\chi \in T^\vee(\mathbb{C}) = \mathrm{Hom}(\Gamma, \mathbb{C}^*) \subset \mathrm{Hom}(T(K_v), \mathbb{C}^*)$

$\mathrm{Ind}(\chi)$ has 1-dim $G(\mathcal{O}_v)$-invariant subspace:
$G(\mathcal{O}_v)$ acts transitively on X flag variety
so any section $G(\mathcal{O}_v)$-invariant is determined by value at single point, det by action of stabilizer on fiber
$\to G(\mathcal{O}_v)$ acts trivially $\longrightarrow$ character trivial on $T(\mathcal{O}_v)$.
$\mathrm{Ind}(\chi)^{G(\mathcal{O}_v)}$ 1-dim - fibers of line bundle (algebraic) over $T^\vee(\mathbb{C})$.

So for any $\mu \in \mathcal{H}(G(K_v), G(\mathcal{O}_v))$ we get function on ~~$\mathbb{C}[$~~ $T^\vee_{\mathbb{C}}$ — the Satake morphism.

Why is image in W-invariant? – intertwining operators.
– conjugate χ by w get "essentially same" representation (at least generic χ)

$\to$ Define canonical morphism $\mathrm{Ind}(\chi) \longrightarrow \mathrm{Ind}(\chi^w)$, isom for generic χ – using integral operators
– kernel on $X \times X$.
$w \in W$ canonical correspondence
– G-orbit on $X \times X$ labelled by w.

$$\begin{tikzcd} & X \times X & \\ & C_w \arrow[u, hook] \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ X & & X \end{tikzcd}$$

Given two Borels - contain common torus, identified with standard T in two ways $\Rightarrow$ automorphism of T, given by w.
$\leftrightarrow G \cdot (1, \text{"}w\text{"} \cdot 1) = C_w$ orbit.
$\to$ intertwiners.

Relative forms on C_w for two projections are naturally dual vector bundles ω_{S/X_1} & ω_{S/X_2}
- naive linear algebra.

$(b_1, b_2) \in C_w$, Tangent to C_w at (b_1, b_2) is $\mathfrak{g}/\mathfrak{b}_1 \cap \mathfrak{b}_2$.

$$\mathfrak{g}/\mathfrak{b}_1 \cap \mathfrak{b}_2 \to \mathfrak{g}/\mathfrak{b}_1, \quad \mathfrak{g}/\mathfrak{b}_1 \cap \mathfrak{b}_2 \to \mathfrak{g}/\mathfrak{b}_2$$

Tangent to images are $\mathfrak{g}/\mathfrak{b}_1$, $\mathfrak{g}/\mathfrak{b}_2$

so rel. tangent bundles are $\mathfrak{b}_1/\mathfrak{b}_1 \cap \mathfrak{b}_2$, $\mathfrak{b}_2/\mathfrak{b}_1 \cap \mathfrak{b}_2$

(vertical spaces)

- generated by opposite roots $\Rightarrow$ dual.

(up to det) - if fix $\langle\,\rangle$ on $\mathfrak{g}$ ...

$S = C_w$ so on det get duality,

$$\Theta_{S/X_1} = \mathfrak{n}_2/\mathfrak{n}_1 \cap \mathfrak{n}_2 \qquad \Theta_{S/X_2} = \mathfrak{n}_1/\mathfrak{n}_1 \cap \mathfrak{n}_2$$

paired by inv quadratic form.

$$\omega_{C_w} = pr_1^* \omega_X \otimes \omega_{C_w/X_1} = pr_2^* \omega_X \otimes \omega_{C/X_2}$$

$$\Rightarrow \quad pr_1^* \omega_X = pr_2^* \omega_X \otimes \omega_{C/X_2}^{\otimes 2}$$

Over set of points: take square roots, $pr_1^* \omega_X^{\frac{1}{2}} = pr_2^* \omega_X^{\frac{1}{2}} \otimes \omega_{C/X_2}$

- measures along fibers

$\to$ can pull back & integrate along fibers.

$\longrightarrow$ Intertwiner

- kernel of intertwining must be in $\mathcal{O} \boxtimes \Omega$ on square

$\leadsto$ must worry about convergence... (might get poles in analytic cont from domain of convergence..)

Finite field case easy - no convergence question...

e.g. Gelfand-Graev for GL_2 over local field.

Beilinson - Introduction II 10/14

$S: \mathcal{H}_v = \mathcal{H}(G(F_v), G(\mathcal{O}_v)) \xrightarrow{\sim} \mathcal{O}(G^\vee_{\mathbb{C}})^{Ad}$

- almost defined over $\mathbb{Q}$: if $\frac{1}{2}$-forms correctly defined,
ie if $\rho = \frac{1}{2}$ sum pos roots is a weight of G, then
S defined over $\mathbb{Q}$ — we had to consider $\sqrt{\text{forms}}$ ...
- In general defined over $\mathbb{Q}(q^{\frac{1}{2}})$, $q = \#$ (resi field)

Global picture S finite set of places of F.
ϕ rep of $G(\mathbb{A})$ $\rightsquigarrow$ invariants $\phi^{\prod_{v\notin S} G(\mathcal{O}_v)}$
($\prod G(\mathcal{O})$'s is maximal compact)
This carries action of each Hecke $\mathcal{H}_v$ $v \notin S$ $\Rightarrow$ $\bigotimes_{v\notin S} \mathcal{H}_v$-module.

$\phi \longmapsto \mathrm{Supp}_S\, \phi \subset \mathrm{Spec} \bigotimes_{v\notin S} \mathcal{H}_v = \prod \mathrm{Spec}\, \mathcal{H}_v = \prod G^\vee_{\mathbb{C}}/Ad$

{really $\mathrm{Spec}\, \mathcal{O}(G^\vee_{\mathbb{C}})^{Ad}$ = semisimple part G^{ss}/Ad}

|| Also have commuting action of
|| $G(F_v)$'s for $v \in S$ — carries multiplicities as well

S-Automorphic spectrum := Supp_S of automorphic forms.

$\alpha: Gal(F, S) \longrightarrow G^\vee_{\mathbb{C}}$ {Unramified outside S $\rightsquigarrow \pi_1^{arith}(X \setminus S)$

[Note $\mathrm{Supp}\, \phi$ only depends on Jordan-Hölder components — can assume irred. Have lots of commuting operators
Multiplicities — count irreducibles]

$\alpha \longrightarrow Fr(\alpha) = (\alpha(Fr_v)) \in \prod_{v\notin S} G^\vee_{\mathbb{C}}/Ad$

S Galois spectrum := $\{Fr(\alpha)\}$ all α.
For $G = GL_n = G^\vee$ no multiplicities

$G^\vee_{\mathbb{C}}$ acts on set of α by conjugation, care only about orbit.
Orbit not completely determined by $Fr(\alpha)$
in general: can have $Fr(\alpha_1) = Fr(\alpha_2)$
for nonconj. α
- Yes for GL_n - rep. determined by character
which amounts to $Fr(\alpha)$ since Frobenii dense
in Galois group. (Chebotarev)

For general G: still use Čebotarev — have α which are pointwise conjugate — are they globally?

Conjugacy classes in GL_n don't generalize nicely to other G. Suppose have two fin groups, two completely decomposable reps into G which are pointwise conjugate — in GL_n they are conjugate, since character defined pointwise ..... not in general

Example (Serre) Group $H = (\mathbb{Z}/3)^2 \longrightarrow G^\vee = PGL_3$.

$$\begin{array}{ccc} \mathbb{C}^* & & \mathbb{C}^* \\ \downarrow & & \downarrow \\ H^{(\alpha)} & \to & GL_3 \\ \downarrow & & \downarrow \\ H & \xrightarrow[\alpha]{} & PGL_3 \end{array}$$

$\left\{ \text{Rep } H \xrightarrow[\alpha]{} PGL_3 \right\} \Longleftrightarrow \left\{ \begin{array}{l} \to \text{central extension} \\ H^{(\alpha)} \text{ of } \mathbb{C}^* \text{ \& rep} \\ H^{(\alpha)} \longrightarrow GL_3 \end{array} \right\}$

Heisenberg extension of H — skew-symmetric form on H values in $\mathbb{C}^*$ for any central extensions

lift generators of H to H^α s.t. $a^3=1, b^3=1$,
$ab = \omega ba \quad \omega \in \mathbb{C}^*$. $\omega^3=1 \quad \omega=\omega(a,b)$
$\omega \neq 1$ Heisenberg extension — 2 nonisomorphic such.
ω uniquely determines the representation: $H^{(\alpha)}$ has unique irrep with center acting as identity — which has dim 3.
Pick any $a \in H$, lift to $H^{(\alpha)}$ with $a^3=1$ — all eigenvalues roots of 1.

$$a \mapsto \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}$$

So both Heisenberg reps are pointwise conjugate — but not equivalent.

$$b \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Exercise: Do such exist for G simply connected?

Galois reps into $G^\vee_{\mathbb{C}}$ have finite image ... to get full Galois picture (as ~~[illegible]~~) replace with ℓ-adic representations.
Really should consider Galois reps + nilpotent element — replace curve by curve times disc

- Pairs (α, N), $N \in \mathfrak{g}^\vee$ so that $\mathrm{Im}(\alpha)$ normalizes line through N: [automatically nilpotent]
$\mathrm{Ad}_{\alpha(g)} N = \chi(g) N$. χ standard cyclotomic character

- i.e. as Galois module N is $\mathbb{Q}_\ell(1)$, Tate module of roots of unity.
- $\mathrm{Gal} \circlearrowright \ell^k$-roots of unity/in $\varprojlim_{k\to\infty} (\mathbb{Z}/\ell^k\mathbb{Z})^* = \mathbb{Z}_\ell^*$

G is over $\overline{\mathbb{Q}}_\ell$ now. Gal profinite topological group, consider only continuous reps. Since $\mathbb{C}$ connected, rep of profinite group factors through group of finite extension.
ℓ-adic case most reps don't factor through finite quotient.

Topology $\mathbb{C}/2\pi i\mathbb{Z} \xrightarrow[\exp]{\sim} \mathbb{C}^*$ $H_1(\mathbb{C}^*) = 2\pi i\mathbb{Z}$
i: complex conjugation acts nontrivially on circle
Finite covers: $\mathbb{C}^*_n \xrightarrow{(\,)^n} \mathbb{C}^*$ $\forall n$ finite covers, try to approximate H_1: $H_1 \longrightarrow \mu_n = \mathrm{Gal}(\mathbb{C}^*_n/\mathbb{C}^*)$
$\ell \longmapsto \exp(\ell/n)$
$\Rightarrow 2\pi i\mathbb{Z} \longrightarrow \varprojlim \mu_n =: \hat{\mathbb{Z}}(1)$ noncanonically isom to $\hat{\mathbb{Z}}$.
$\_(1) \longleftrightarrow 2\pi i$.
$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ etc all act on $\hat{\mathbb{Z}}(1)$, but not on $H_1(\mathbb{C}^*)$ — acts by cyclotomic character.

$\mathbb{C}^* \rightsquigarrow E$ elliptic curve $\Rightarrow$ 2 dim representation.
n-torsion $\simeq \mathbb{Z}/n \oplus \mathbb{Z}/n$ $\rightsquigarrow$ in limit something $\rightsquigarrow \hat{\mathbb{Z}}^2$
Tate module of E. — order ℓ^k pts $\rightsquigarrow \mathbb{Z}_\ell$-rep.

N in function field case: have $G^\vee$ local system on $X - S$, now replace X by $X \times D^\times$, local systems which are purely unipotent in disc direction, ~~dise~~ N is log of monodromy — arithmetic fund group is not ~~semi direct~~ product, identify two Frobenii

pt $\times$ pt $=$ pt but with fund. groups is just Frob ...

Purity for (α, N): should hold automatically in irred reps, G semisimple.

Want purity weight zero.

$N=0$: this means Fr elements should have abs value 1 ($\longleftrightarrow$ unitary reps on automorphic side) (- after rep $G^\vee \to GL_n$)

$N \neq 0$: Pass to GL_n; local system behaves wrt weights exactly as nearby cycles for pure local system on surface extending $X \times D^*$ $\times S$

Exerts $N \leadsto$ Lefschetz operators from Shimura variety point of view, on cohomology (forms of different weights?)

Simplest example: trivial rep $\leadsto$ autom forms - on Galois side need something nontriv (triv $\leadsto$ Eisenstein series) $\to$ assign α acting on N principal nilpotent as diagonal element of principal $\mathfrak{sl}_2$.

Counting Gal reps with symmetries conjecture correspondence of automorphic $\longleftrightarrow$ Galois spectra (with multiplicity) - in fun field side would show Galois side indep of ℓ, since automorphic is.

Number field case much more subtle: even $SL(2, \mathbb{A})$: consider reps with $\neq 0$ vector invariant wrt $SO(2) \times SL(2, \hat{\mathbb{Z}})$ maximal compact.

$SL(2, \mathbb{Z}) \circlearrowright \mathbb{H} = SL_2\mathbb{R}/SO(2)$

Spectrum: want $L^2(SL_2\mathbb{Z} \backslash \mathbb{H}) \circlearrowleft T_p$ for all p Hecke

$\circlearrowleft \Delta$ Laplacian (~ "T_∞")

- Continuous & discrete spectrum for Δ, consider discrete part.

T_p come from $\mathcal{H}(SL(2, \mathbb{Q}_p), SL(2, \mathbb{Z}_p)) \simeq \mathbb{C}[z, z^{-1}]^{\mathbb{Z}/2}$

$T_p \longleftrightarrow$ generator $z + z^{-1}$. $= \mathbb{C}[z+z^{-1}]$

Fix discrete eigenvalue of Δ (conj. mult $= 1$) $\Rightarrow$ function with T_p eigenvalues — should equal Frobenii for some ℓ-adic rep: but these are ℓ-adic numbers while on modular function these are real numbers — need to work to compare — e.g. need these eigenvalues algebraic. — not true for all examples — eg real quadratic extensions, Hecke character nontrivial on real part..... don't know anything about these!

Part of our stupidity? should have hidden part of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ — maybe have other parts with strange reps...

No such formulation by Langlands — should have "true" Galois group & reps — formulate instead as functoriality principle:

Suppose $\rho: Gal \to GL(n) \to GL(N)$

$\Rightarrow$ should have similar construction on automorphic side:

$GL(n)$ automorphic rep $\rightsquigarrow GL(N)$ automorphic rep, so that $GL(N)$ spectrum expressed explicitly in terms of $GL(n)$ spectrum. $\longrightarrow$ predictions which make sense for number fields.

For GL_2 proven for some symmetric powers of basic rep but not symplectic: get fns on $SO(n)\backslash SL(n,\mathbb{R})/SL(n,\mathbb{Z})$

Bernstein: Don't know how to compare ℓ-adic sheaves for different ℓ... Parallel: construct $\mathbb{R}$ through Dedekind cuts, wrong way to think .....

Class field theory Classes $F^\times\backslash \mathbb{A}^\times/\mathcal{O}^\times$ (ideles) $\simeq$ max unram abelian quotient of Gal group.

$\mathbb{A}^\times/F^\times \sim \mathbb{Z} \times$ compact grp

Decompose functions on $\mathbb{A}^\times/F^\times$: unitary characters of $\mathbb{A}^\times$ trivial on $F^\times$.

$Gal/[Gal, Gal] \simeq \mathbb{A}^\times/F^\times$. More precisely: $Gal^{ab} =$ Profinite completion of $\mathbb{A}^\times/F^\times$

Function field case: $\mathbb{A}^\times/F^\times$ is completely discont, so passing to profinite completion is innocent. BUT number field case have continuous parts — eg real quad extensions involving log(basic unit)

— know $Gal/[Gal,Gal]$ quotient of "true" object $\mathbb{A}^\times/F^\times$.

But don't see $\mathbb{A}^\times / F^\times$ itself as automorphisms of something – mysterious, like eigenvalue question.

<u>Geometry & Combinatorics</u> 10/21

(<u>Beilinson</u>) $\mathcal{H}(G(F_v), G(O_v)) \xrightarrow{\sim} \mathcal{O}(G^\vee)^{Ad} \xleftarrow[\text{char}]{\sim} R(G^\vee)$

functions with compact support, $G(O)$-invt on $Gr_v = G(F_v)/G(O_v)$ — $R(G^\vee)$: basis of irreps

Not just set-combinatorial structure: Bruhat-Tits building

$G = GL_n = GL(F_v^n)$. $L \subset F_v^n$ lattice: f.g. O_v-submodule which generates F_v^n as vector space / F_v – any such generated by some basis in F^n under O_v. $\rightarrow Gr_v$ = set of lattices in F_v^n.

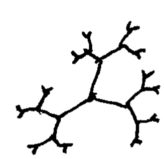
SL_n: def of lattice is lattice in F $\rightarrow \mathbb{Z}$-torsor ($\mathbb{Z} = F_v^*/O^*$) $\Rightarrow$ integer "additive volume" ($q^{\mathbb{Z}}$ gives real volume of normalized measure). SL_n acts on lattices of additive volume zero (mult. vol 1).

PGL_n: lattices modulo homotheties

$Gr^{SL_n} \hookrightarrow Gr^{GL_n} \twoheadrightarrow Gr^{PGL_n}$! homotheties can change volume only mod n $\rightarrow$ Gr^{SL_n} corresponds to vol $\equiv 0$ (n) in Gr^{PGL_n}.

PGL_2: Given two lattices mod homotheties L_1, L_2, they are <u>neighbors</u> if (say) $L_1 \subset L_2$, $L_2/L_1 \simeq k$.
Up to homothety this is actually <u>symmetric</u>: $L_1 \subset L_2 \subset \pi^{-1} L_1$

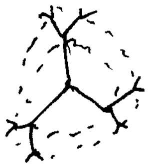
Vertices $\longleftrightarrow Gr^{PGL_2}$, edges $\longleftrightarrow$ neighbors.
$\rightarrow$ neighbors of fixed L $\simeq \mathbb{P}^1(k)$ $\quad k = \mathbb{Z}/2$
Claim – this is tree
[distance function: length of quotient module after arranging inclusion]



Relation to $\mathbb{P}^1(F_v)$ – compact (profinite) topological space
– $\mathbb{P}^1(F_v)$ = <u>ends</u> of our tree: outgoing paths up to equivalence $\longleftrightarrow$ all paths from fixed initial point, "close to deg n" paths – agree for first n steps
$\Rightarrow$ profinite topology

Construction of map: Given $F_v^2 \supset R$ line
→ construct path $L,\ L+\pi^{-1}(L\cap R),\ L+\pi^{-2}(L\cap R)\dots$

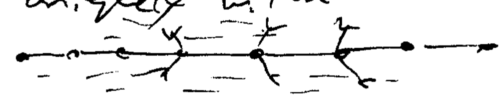
→ make L more & more like R ... ⇒ homeomorphism.



Orbits $PGL_2(O_v)$: stabilizer of point, preserves distance ⇒ circles
⇒ Hecke: compactly supported fns of radius

Unipotent $T(F_v)$, $N(F_v)$

$T(F_v)$: preserves two lines in F_v ⇒
two paths on tree ⇒ connect uniquely without
repetitions into biinfinite path on tree



— translates along tree.
So T preserves distance from path
→ orbits labelled by distance

$N(F_v) \subset B(F_v)$: stabilize unique line
→ sequence of points on tree moving to ∞. Compactify tree
by adding $\mathbb{P}^1(F_v)$: this is limit of B-stabilizers
in finite points ... — distances form $\mathbb{Z}$-torsor
now: differences between ∞ finite points.

Given two points in tree: connect both (uniquely)
to same point at ∞ ⇒ well defined
difference of distances to ∞.

$T \curvearrowright N\backslash Gr$ factors through $T(F)/T(O) = \mathbb{Z}$
— gives $\mathbb{Z}$-torsor structure.

General G: $T(O)\backslash T(F)/W = \Gamma/W \xrightarrow{\sim} G(O_v)\backslash Gr$

$N\backslash Gr$ N orbits ⇝ $T/T(O_v) = \Gamma$.

(Drinfeld) $SL_2(F)/SL_2(O) \hookrightarrow PGL_2(F_v)/PGL_2(O_v)$
What is image in tree: Color the tree
even/odd, image consists of all even vertices

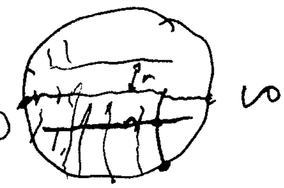
Shape of orbits: $PGL_2(O_v)$: first circle $\simeq \mathbb{P}^1(k)$
Circle of radius n $\simeq \mathbb{P}^1(O_v/m_v^n)$
Points at ∞ $\simeq \mathbb{P}^1(F_v) = \mathbb{P}^1(O_v) \xrightarrow{\text{maps}}$ circle of radius n
— given pt at ∞ → $\mathbb{P}^1$ compact ⟺ regular

find unique pt of given circle of radius n.

$\longleftrightarrow$ map $\mathbb{P}^1(\mathcal{O}_v) \to \mathbb{P}^1(\mathcal{O}_v/m_v^n)$.

$\rightsquigarrow$ line in n dim module for two L of distance n.

Fix torus $\longleftrightarrow$ fixing two points at ∞



$\mathbb{P}^1 \setminus \{0,\infty\} \cong F_v^* \longrightarrow \{\text{orbit number } n\} \cong F_v^*/U_n$

$U_0 = \mathcal{O}_v^*$ $n \geq 0$ $U_n = 1 + m_v^n$ $(n>0)$

N orbits:

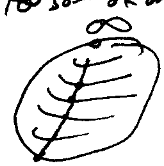
$\mathbb{P}^1(F_v) \setminus \{\infty\} = F_v$ $(\mathbb{A}^1)$

$\mathbb{Z}$-torsor of orbits

$\downarrow$

Orbit number n

$\cong F_v/m_v^n$



- we've already fixed another point at ∞ to identify $\mathbb{P}^1 \setminus \{\infty\}$ with F_v: really affine torsor....

$B(F_v)$ changes n but $N(F_v)$ fixes n

Hecke algebra: Function on vertices depending only on distance, finite support. What about algebra structure?

Compact-supp $C_0(PGL(2,F_v)/PGL(2,\mathcal{O}_v))$: generated by fns at vertices

$\mathrm{End}_{PGL_2(F_v)} C_0 = \mathcal{H}(PGL(2,F_v)/PGL(2,\mathcal{O}_v))$ always true for any pair (G,K).

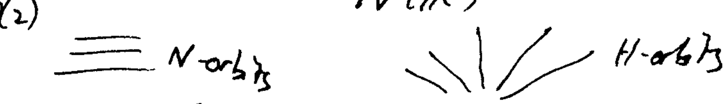
- $\mathrm{End}_G C_0^\infty(G/H) = \mathcal{H}(G,H)^{opp}$ H compact, open - at least..

First circle: $(Tf)(x) = \sum_{y \text{ neigh}(x)} f(y)$

Analogy with $\mathbb{R}$: $PGL(2,\mathbb{R}) \supset H(\mathbb{R})$

$O(2) \subset$ $\supset N(\mathbb{R})$

$PGL_2\mathbb{R}/O(2)$



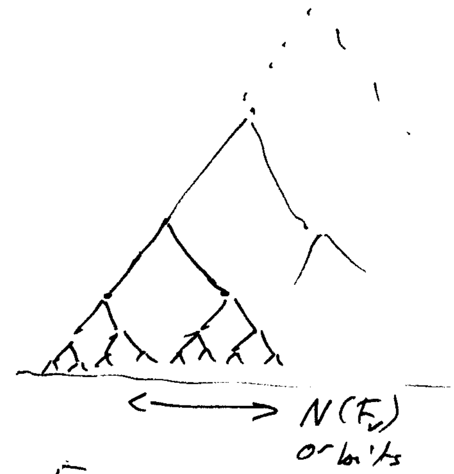
$O(2)$: see as

- distinguished pt in interior in disc picture

$\Delta f = 0 \iff$ average of f on disc is f

$\longleftrightarrow$ $Tf = (q+1)f$.

"Upper half plane" picture of building:



(Beilinson) <u>Geometry</u>

k residue field,

Gr_v forms k-points of a space over k.

$Gr_v(R)$ [R comm. k-algebra]

namely $G(F_v \hat{\otimes} R)/G(O_v \hat{\otimes} R)$

$F_v \hat{\otimes} R = R((t))$.. Better — assume R <u>local</u> ring (localize in Zariski topology)

$G(F_v)(R)$, $G(O_v)(R)$ also group-spaces.

- don't need to sheafify — already sheaves.

$G(O_v)$ scheme, affine : projective limit of $G(O/m_v^n)$

- jets of points of G.

Assume $G = \mathbb{A}^1$ for simplicity:

$\mathbb{A}^1(O_v)(R) = R \hat{\otimes} O_v = \varprojlim R[t]/t^n R[t] \simeq R^\infty$ as set $R \times R \times \dots \times R$

represented by scheme $\mathbb{A}^\infty = \mathrm{Spec}\, k[t_1, t_2, \dots]$

X affine algebraic variety $\rightsquigarrow X(O_v)$ also affine scheme

- represent as closed subvariety of $\mathbb{A}^n \to$ closed subvariety of $X(O_v)$.

{Affine schemes} closed under $\varprojlim$ $\Leftrightarrow$ $\varinjlim$ algebras,

$X(F_v)(R) = X(F_v \hat{\otimes} R)$. $\mathbb{A}^1(F_v)(R) = R((t)) = \bigcup t^{-n} R[[t]]$

$\Rightarrow \mathbb{A}^1(F_v) = \varinjlim \mathbb{A}^1(O_v) \xrightarrow{t} \mathbb{A}^1(O_v) \hookrightarrow \dots$

Result not affine - not quasicompact.. in fact not scheme.

- ind-affine scheme Union of schemes wrt closed embeddings

- but have others which aren't given by closed embeddings!

Gr : can embed into larger space which is obviously ind-scheme

Fix lattice $L_0 = O_v^n \subset F_v^n$.

$t^N L_0 \subset L \subset t^{-N} L_0$ for any L some N, O_v-submodule

$\longrightarrow$ scheme O_v-submodule of $t^{-N}L_0 / t^N L_0$

Now look at larger space: just k-vector subspaces of F_v that sit as above (commens. with L) — only depends on linear topology of F_v, basis of nbhds given by lattices

$\Rightarrow Gr^{Sato}(F_v^n) = \bigcup_N Gr(t^{-N}L_0/t^N L_0)$ (arbitrary dim)

Gr_v just $\mathcal{O}_v$ fixed points $\leftrightarrow [Gr^{Sato}]^{1+t}$ 1+t fixed points gives correct answer on functor level as well.

Gr^{Sato} very homogeneous — but for any given piece, subscheme $\subset Gr_v$ got singularities — looks singular at every level but still formally smooth (homog. space)

If no torus quotient $\Rightarrow$ reduced. (every "piece" bad but reduced)

"Reduced" means ind-reduced.

$G(\mathcal{O}_v)$-orbits: all have same parity of dimension on given component.

PGL_2: two components ($\leftrightarrow$ 2-coloring of graph)

- edges in graph always connects two different connected components! has to do with correspondences, not topology.

Circle radius 1 different color, but circle radius two has one point in the closure

closure is quadratic cone on $\mathbb{P}^1$, vertex the zero-dim orbit! $Gr(2,4)$ in Sato $(N=2)$

$Gr_2(t^2L/t^2L)$ fixed wrt unipotent $1+t$ $\begin{pmatrix} \boxed{1} & \\ & \boxed{1} \end{pmatrix}$

$\simeq \begin{matrix} \mathcal{O}(-2) \\ | \\ \mathbb{P}^1(k) \end{matrix} \Rightarrow$ cone.

[Beilinson, cont.] $G(\mathcal{O}) \circlearrowright Gr = G(K)/G(\mathcal{O})$

finite dimensional orbits (Gr ind-finite type, can choose pieces $G(\mathcal{O})$-invariant). labelled by $\Gamma^+ \subset \Gamma = H(K)/H(\mathcal{O})$

positive ~~dominant~~ weights for ${}^LG^\vee \longleftrightarrow \mathrm{Irr}\, G^\vee \longleftrightarrow \Gamma/W$.

$\chi \in \Gamma_+ \longmapsto \mathrm{Orb}_\chi$ "sphere of radius χ"

$N(K)$-orbits: "horospheres" $\longleftrightarrow \Gamma \ni \gamma \leftarrow \tilde{\gamma} \in H(K)$

$H \subset G \Rightarrow S_\gamma = N(K)\cdot \tilde{\gamma} \subset Gr$.

$\mathrm{Orb}_\chi = G(\mathcal{O})/G(\mathcal{O}) \cap \tilde{\chi} G(\mathcal{O}) \tilde{\chi}^{-1}$ $\tilde{\chi} \in H(K) \subset G(K)$ representative of χ.

$1 \to \{\text{prounipotent}\} \to G(\mathcal{O}) \to G \to 1$

$1 \to$ n-jets to $G \to \varprojlim G(\mathcal{O}/m^n) \to G \to 1$

GL_n: $\begin{pmatrix} t^{k_1} & & \\ & \ddots & \\ & & t^{k_n}\end{pmatrix}$: $GL_n \circlearrowleft$ $a_{ij} \mapsto t^{k_i - k_j} a_{ij}$

Positive element $(k_1 > \dots > k_n)$ two Borels B_+, B_- get scaled in opposite directions. $G(\mathcal{O}) \cap \tilde{\chi} G(\mathcal{O}) \tilde{\chi}^{-1}$ will contain one of the Borels & part of the other (on finite G level) — maps to parabolic in G, with some prounipotent quotient (orderings of k's $\leftrightarrow$ parabolics).

Thus $\mathrm{Orb}_\chi \longrightarrow G/P$ smooth fibration, fibers are quotients of unipotent groups $\Rightarrow$ affine spaces: affine bundle over projective variety.

<u>Actually</u> has <u>section</u>:

* $\mathrm{Aut}\,\mathcal{O} \circlearrowright Gr$ $\mathcal{O}$ = formal power series.

proalgebraic, extension of $\mathbb{G}_m$ by prounipotent.

$t \mapsto \varphi(t)\cdot t$ $\varphi(t) = a_0 + a_1 t + \dots \in \mathcal{O}^*$

$\to$ ~~extension of $\mathbb{G}_m$~~ .. $\mathcal{O}^*$ has filtration & this is <u>group</u> isomorphism on <u>graded</u> level.

$t \mapsto a_0 t$ $\mathbb{G}_m$ action $\Rightarrow$ action on Gr. preserves orbits Orb_χ. [once we pick a parameter]

$\mathbb{G}_m$ Fixed points: on Orb_χ get copy (section) of G/P $\Rightarrow \coprod_\chi G/P_\chi$. On affine fibers $\mathbb{G}_m$ acts with positive weights — contracts orbit on fixed points.

Equivalently take $G \cdot \tilde{\chi} G(G)$ orbit of our representative is G/P.

$H(X)/H(G) \subset Gr$ plays same role as standard set of axes in $\mathbb{P}^n$ - e.g. distinguish sections of line bundles on $\mathbb{P}^n$ by their values on these "axis" points.

- Note: all orbits are simply connected (homotopic to G/P)
- Parity of dimensions constant on connected components of Gr.
  $\dim Orb_\chi = 2\langle \rho, \chi \rangle$ $\quad \rho = \frac{1}{2}$ sum of positive roots.
  thus if ρ is a weight of $G \Rightarrow$ all orbits have even dimension.
- Connected components of $Gr \longleftrightarrow \pi_1(G)$
  via loop $\mapsto$ its class in π_1 (over $\mathbb{C}$), or algebraically
  $\tilde{\chi} \in \Gamma \twoheadrightarrow \Gamma/$ one-param coroots
  Principal SL_2 in $G^\vee$ (Cartan elt $\mapsto \rho$) distinguishes orbits by even/odd weight...
  $Orb_\chi, Orb_{\chi'}$ lie in same connected comp $\Leftrightarrow \chi - \chi'$ lies in coroot lattice
  $\Leftrightarrow$ same value on Center $(G^\vee) = \pi_1(G)^\circ$ dual.

Dimensions of S_γ form $\mathbb{Z}$-torsor: by differences of dimensions of stabilizers of points: have finite codimensions ... differences look like dimensions of Orb_χ.

Return to Finite Fields:

Fonctions $\longleftrightarrow$ Faisceaux $\qquad X/\mathbb{F}_q, \quad X(\mathbb{F}_q) \subset X(\mathbb{F}_{q^2}) \subset \dots \subset X(\overline{\mathbb{F}}_q)$

$Fr_q \in Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q) \circlearrowright X(\overline{\mathbb{F}}_q)$

$Fr_q^n = Fr_{q^n}, \quad X(\mathbb{F}_{q^n}) = X(\overline{\mathbb{F}}_q)^{Fr_q^n}$.

$Fun(X) = \{ (f_1, f_2, \dots) \quad f_i : X(\mathbb{F}_{q^i}) \longrightarrow \overline{\mathbb{Q}}_\ell \}$

Interesting functions: assume $\mathcal{F}$ constructible sheaf on X.
In usual topology - sheaf of fin dim vector spaces, s.t. $\exists$ finite stratification s.t. $\mathcal{F}|_{stratum}$ locally constant

étale finite sheaves $\rightsquigarrow \mathbb{Q}_\ell$ constructible sheaves...

$x \in X(\mathbb{F}_{q^n})$, $\mathcal{F}_x$ carries Frobenius action (x Fr_{q^n} fixed pt $\Rightarrow$ Fr_{q^n} acts on fiber).

$\{Tr(Fr^n, \mathcal{F}_x)\} \in Func(X)$ (notice not same on $X(\mathbb{F}_q)$ & $X(\mathbb{F}_q) \subset X(\mathbb{F}_{q^2})$ - different power).

- same spectrum, not same trace!
- these & their linear combinations give subspace $\subset Func(X)$

Čebotarev density $\Rightarrow$ the function "determines" the sheaf - rather the class of $\mathcal{F}$ in $K(\bar{\mathbb{Q}}_\ell\text{-Sheaves on }X)_{\bar{\mathbb{Q}}_\ell}$!!

$$\int$$

(certainly any function on finitely many points comes from a sheaf, built by skyscrapers — nontrivial structure is global...) $Func(X)$

More natural objects - perverse sheaves .. good objects & truly exist with no good definition

Usual sheaves (constructible) have infinite length — e.g. const. sheaf on curve: get subsheaf for every point, get descending chain of infinite length

- Perverse sheaves have finite length: corresponding perverse sheaf is irreducible: deleting points get larger sheaves...

Irreducible objects: take local system on Zariski open of closed $Y \subset X$ and extend it $(\ )_{!*}$ — follows from finite length hypothesis... also local objects.

Goresky-MacPherson

$U \overset{open}{\subset} Y \overset{closed}{\subset} X$ assume have fn on U coming from local system $\Longrightarrow$ canonical extension of the function to Y, via perverse sheaves. (If closure smooth get just constant).

Case of curve $X \supset U = X \setminus \{x_1 .. x_n\}$: extend to x_n by invariants of local monodromy (naive sheaf pushforward). Higher dim - this works for codim 1,2 but higher need truncations..

$\boxed{\text{Problem}}$: in algebraic geometry get only fin dim objects

— no way to truly deform localsystems (except infinitesimally) algebraically ... complex situation $\rightsquigarrow$ $\mathcal{D}$-modules. May consider families — since know about non-holonomic $\mathcal{D}$-modules!

Satake $s: \mathcal{H}(G(K), G(\mathcal{O})) \xrightarrow{\sim} \mathcal{O}(G^\vee)^{Ad} \xleftarrow[tr]{\sim} R(G^\vee)$
$\cup$ irrep basis Irr_χ

$k = \mathbb{F}_q$. Take constant 1 on Orb_χ & take its G-M extension: $Tr_{Fr}(IC_{orb_\chi})$ (up to normalization by dim of orbit $q^{2\langle \rho, \chi \rangle}$) $\longleftrightarrow Irr_\chi$ (Lusztig).

In fact $P_{G(\mathcal{O})}(Gr) \xrightarrow{\sim} Rep(G^\vee)$, lifts above on K-group.
[Lusztig: RHS is semisimple, know its irreducibles $\to$ implies identification... but actually have canonical identification]
- equivalence of tensor categories (tensor on LHS lifts convolution on $\mathcal{H}(G(K), G(\mathcal{O}))$.
— nontrivial part is that convolution is perverse sheaf, not complex!

Geometric automorphic forms:
Usual theory: functions on $G(\mathcal{O}_\mathbb{A}) \backslash G(\mathbb{A}) / G(F)$
(unramified case). $F = k(X)$ "$Bun_G(k)$" set of isom classes of G-bundles on X (Weil)

$G(\mathbb{A}) \longleftrightarrow (\mathcal{F}, \gamma_\eta, \{\gamma_x\})$
$\{\gamma_x / \gamma_\eta\}$ generic triv formal trivs

Passing to isomorphism classes - breaks e.g. sheaf structure
Perverse sheaf on Bun_G provides not just function on $G(\mathcal{O}_\mathbb{A}) \backslash G(\mathbb{A}) / G(F)$ (automorphic form) but also all its liftings to extensions $\mathbb{F}_{q^n}$ of our field ...

Hecke eigensheaves "Eigenvalues" are $G^\vee$-local system on X. $x \in X$

$$\begin{tikzcd} & Hecke_x \ni (\mathcal{F}_1, \mathcal{F}_2, \nu) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{F}_1 \in Bun_G & & Bun_G \ni \mathcal{F}_2 \end{tikzcd}$$

$\nu : \mathcal{F}_1 /_{X \setminus x} \xrightarrow{\sim} \mathcal{F}_2 /_{X \setminus x}$.

Our first factor: fibration, with fibers versions of Gr..

Trivialize $\mathcal{F}_1, \mathcal{F}_2$ near x. to modify $\mathcal{F}_1$ to $\mathcal{F}_2$ need element of $G(K)$ $\quad \nu^{-1}(\gamma_x^2)/\gamma_x^1 \quad$ (γ_x^i trivs)

(changing $\gamma_x^2 \longleftrightarrow$ element of $Gr_x(k)$.

Don't pick $\gamma_x^1 \Rightarrow$ twisted form of Gr.

By twisting, $P_{G(O)}(Gr)$ acts as kernel on sheaves on Bun_G. $IC_{\overline{Gr}_\lambda}$ on level of functions gives classical Hecke operator

$$\phi \in P_{G(O)}(Gr_x), \quad S(\phi) = V \in Rep\, G^\vee$$

$$\circlearrowright D(Bun_G) \ni ? \mapsto p_{2*}(\phi \otimes p_1^*(?)) = \phi_x(?)$$

$?$ is a Hecke$_x$ eigensheaf if for each ϕ, $\phi_x(?) = S(\phi) \otimes ?$
as vect space

Better: on RHS want not just vector space but one with an operator $Fr_x \in G^\vee$ & ϕ compatible with Frobenius action on x. On functions get Hecke eigenfunction, $Fr_x \rightsquigarrow Tr_{S(\phi)}(Fr_x)$.

Vary x

$$\begin{tikzcd} & Hecke \arrow[dl] \arrow[dr] & \\ Bun_G & & Bun_G \times X \end{tikzcd}$$

- instead of data of $Fr_x \in G^\vee$ want actually $G^\vee$-local system ψ on X

$\to$ demand $\boxed{\phi(?) \simeq S(\phi)_\psi \boxtimes ?}$ $\quad ? \in Ob\, D(Bun_G, \overline{\mathbb{Q}}_\ell)$, $\psi \in Ob\, LocSys_{G^\vee}$

(in truth should give compatible system of such isomorphisms).

- definition good for perverse sheaves, but this is only core of a putative triangulated category...

Hopefully, when ψ is irreducible, $?$ should be <u>perverse</u> sheaf ... all understood ones are ....