A. Beilinson - Nearby Hecke Functors & Kac-Moody Modules 6/24/00

$\mathfrak{g}$ semisimple, $K = k((t))$ ⇝ $\mathfrak{g}(K)^\wedge$ central extension, study modules so that $\mathfrak{g}(K)^\wedge$ acts continuously (module has discrete topology). Study similarities with p-adic groups ...

Bernstein center: according to local Langlands philosophy, has to do with functions on $Gal(K) \longrightarrow {}^L G$ .. (local systems on Spec K.)

deRham version: moduli $\mathcal{LS}_{{}^L G}$ of local systems for ${}^L G$ on Spec K.

⇝ Bernstein center should relate to functions on $\mathcal{LS}_{{}^L G}$.

Consider representations with level integral and < critical.

∗ Both Bernstein center of this category & functions on $\mathcal{LS}$ are trivial !!

H-C for f.d. $\mathfrak{g}$ $Z(U\mathfrak{g}) \cong \mathbb{C}[\mathfrak{h}^*]^W$

· Verma modules depending on parameter give functions on here, intertwiners give invariance, this gives the isomorphism.

for $\hat{\mathfrak{g}}$: elt of center gives function on dual to Cartan, & get $\boxed{W_{aff}\text{-inv't}}$ functions, noncritical ⇒ lattice acts by translations ⇒ no such functions!

Consider bundles with regular singularity: up to unipotents come from torus. Consider residue of the connection, element of same space as before, But gauge transformations give action of lattice ⇒ again no invariants.

So triviality for same reasons in both cases!

⇒ try to reformulate.

• Category has spectral decomposition wrt center - can localize over Spec (center) "spectral thm for rep category".

$$\boxed{\mathfrak{g}(K)^\wedge\text{-mod has a "spectral decomposition" wrt } \mathcal{LS}_{{}^L G}}$$

What does $\mathcal{LS}_{{}^L G}$ look like? - Can consider families of local systems - structure of quotient of ind affine indscheme of ind finite type by an ind proper equivalence relation.

"ind": union of schemes wrt family of closed embeddings.

Example: line bundles $\mathcal{LS}_{\mathbb{G}_m}$. Consider trivialized line bundles

- described by the "potential" $\in \Omega(K)$, modulo gauge equivalence by K^*.
  - depends only on polar part:
    $\Omega(K)/K^* = (\Omega(K)/\Omega(\mathcal{O})) / (K^*/\mathcal{O}^*)$
    ind affine (vector space)

$K^*/\mathcal{O}^*$: $\mathbb{Z}$ naively ... but allowing parameters get
$\mathbb{Z} \times (K/\mathcal{O})^\wedge$ formal group of $K/\mathcal{O}$.

$\Omega(K)/\Omega(\mathcal{O})$: $k \times d(K/\mathcal{O})$
(first order poles) (exact)

$\mathbb{Z}$ acts by translations on k, $(K/\mathcal{O})^\wedge$ acts by translation on $d(K/\mathcal{O})$. — vector space / its formal group...
Quasicoherent sheaves on this: representations of Heisenberg system $(K/\mathcal{O} \times \mathcal{O})^\wedge$ $\mathcal{D}(K/\mathcal{O}$
$\mathbb{Z} \in \mathcal{O}$ translations, $K/\mathcal{O}$ give vector fields!
get "Heisenberg system". ... typical noncommutative geometry.

Don't ~~know~~ similar description for general G — only know its points! irregular part comes from some torus after finite cover .. ~~get~~ collection of components looking like commutative case, but intersect in complicated ways!

Rough definition of spectral decomposition: For R comm algebra,
φ : Spec $R \to \mathcal{L}\mathfrak{g}_G$ R-point
$\rightsquigarrow$ assign a category of R-modules with $\mathfrak{g}(K)^\wedge$-action, functorial wrt R (should relate to map $R \to$ Bernstein center in usual setting..)
Ind-setting: should twist R-modules to different t-structure! replace pullback$()^*$ by $()^!$...

Given p-adic rep

p-adic setting: have Satake isom for unramified reps & global picture: for any rep of Gal try to extend to global setting: try to extend to automorphic rep of full group $\rightsquigarrow$ global Galois rep ... can look back at local components.

Kac-Moody setting: can consider ~~living~~ on "larger" disc than p-adic setting ... don't need global automorphic setting ..

- Have Satake in deRham setting
- VA structure: can reproduce Hecke action.

<u>Satake picture</u> spherical Hecke algebras $\longleftrightarrow$ Rep ${}^L G$.

Rep ${}^L G \rightsquigarrow$ perverse sheaves on Gr. $\quad \underset{\text{Sheaves}}{tr(Fr \text{ on } \mathbb{I})} \longleftrightarrow$ irred

Pass to $\mathcal{D}$-modules, twisted by line bundles $\longleftrightarrow$ level

so global sections are $\mathfrak{g}(K)^\wedge$-modules.

$\Rightarrow$ Functor Rep ${}^L G \longrightarrow \mathfrak{g}(K)^\wedge$-mod

On isomorphism classes: multiply h.w. of $V \in$ Rep ${}^L G$ by level, get weight for G, take irred h.w. of $\mathfrak{g}(K)^\wedge$

-- get very small reps this way.

(level: invariant quadratic form, $\mathfrak{h} \to \mathfrak{h}^*$, identifies weights)

These reps $\longleftrightarrow$ $\mathcal{D}$-mod on $G(K)/G(O)$ inside $G(K)/I$ $\mathcal{D}$-mod.

get extra relations .. $G(O)$ integrable, & many more relations

Negative level: Vermas have finite length

Positive: " " " colength.

<u>Hecke functors</u> $V \in$ Rep ${}^L G$, $\mathcal{H}(V)$ corresponding $\mathfrak{g}(K)^\wedge$-module

$x \in X$ curve ~~infs~~ $K_x \subset X$, $\mathcal{H}(V)_x$ corresponding $\mathfrak{g}(K_x)^\wedge$-mod

x varies: get $\mathcal{D}$-module on X.

x parameter on X $\Rightarrow$ Lie algebra $\mathfrak{g}[x, t]_{(t-x)^{-1}}$ localize by $(t-x)$

.. get action compatible with connection on everything.

$M \in (\mathfrak{g}(K_0)^\wedge\text{-mod})$, $0 \in X$, $K = K_0$. $U = X \setminus 0 \overset{j}{\hookrightarrow} X$

$M \otimes j_* \mathcal{H}(V)$ (poles at 0) $\mathcal{D}$-mod,

has action of $\mathfrak{g}[x, t]_{(t-x)^{-1}, t^{-1}}$

$$\mathfrak{g}[x, x^{-1}]((t)) \circlearrowright M \qquad \qquad \mathfrak{g}[x, x^{-1}]((t-x)) \circlearrowright \mathcal{H}(V)$$

Consider maximal $\mathcal{D}$-mod quotient of $M \otimes j_* \mathcal{H}(V)$ supported at x & invariant wrt this Lie algebra action.

$\Rightarrow$ action of $\mathfrak{g}[x, t]_{(t-x)^{-1}, t^{-1}}$ mod $x = \mathfrak{g}[t, t^{-1}]$

which is continuous ... $\Longrightarrow$

get functor $M \longmapsto \mathcal{H}\ell(V)(M)$ = this vector space with $\mathfrak{g}(X)$-action

V = trivial rep $\Rightarrow$ identity functor

Can tensor by local systems : $\mathcal{L}$ loc sys on Spec K, $\mathcal{H}\ell(V)_{\mathcal{L}}$ defined in same way but start with $M \otimes \mathcal{H}\ell(V) \otimes \mathcal{L}$ -- control all possible ramification.

Rough property of compatibility of spectral decomposition with the $\mathcal{H}\ell(V)$:

M comes from category of reps corresp to $\varphi \in R\text{-}pt(\mathcal{L}\mathfrak{J}_{\check{G}})$

Want $M \longrightarrow \mathcal{H}\ell(V)_{V\varphi}^{*} M$

system of such with compatibilities

For each loc sys consider fiber of category : class of modules over it.. ~~sta~~

If loc sys has regular singularities :

should be $\mathcal{D}$-mod on affine flags with all possible Cartan twistings..

Given a rep, take its support on $\mathcal{L}\mathfrak{J}_{\check{G}}$ that reduce to tori over K not an extension :

restrict module to $\mathfrak{n}(K)$, take $\infty/2$ cohomology

$\rightarrow$ module over Heisenberg of tori, consider support over $\mathcal{L}\mathfrak{J}_{Cartan}$ & this should give support.

Bezrukavnikov : this picture for purely unipotent local system.