A. Beilinson – Triangulated Categories II 11/19/01

Definition Triangulated Category: D additive category, with

Data
- autoequivalence $D \xrightarrow{\sim} D$ $\quad X \mapsto X[1]$
- class of diagrams in D of shape $X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{p} X[1]$ called distinguished triangles

Properties

Tr 1:
- class of dist triangles closed under isomorphism
- $X \xrightarrow{id} X \to 0 \to X[1]$ is distinguished all X
- Any $X \xrightarrow{f} Y$ can be included in a distinguished Δ

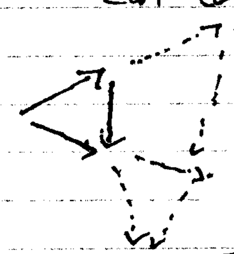
(any C arising this way is called a cone of f)

Tr 2 Rotation: $X \to Y \to C \to X[1]$ distinguished
$\implies$ $Y \xrightarrow{i} C \xrightarrow{p} X[1] \xrightarrow{-f[1]} Y[1]$ is distinguished

Tr 3
$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & C & \longrightarrow & X[1] \\ a\downarrow & & b\downarrow & & \vdots c & & \downarrow a[1] \\ X' & \longrightarrow & Y' & \longrightarrow & C' & \longrightarrow & X'[1] \end{array}$$
$\exists c$ making all diagrams commute

– every morphism between arrows can be extended to cones (not uniquely..)

Tr 4 Can complete commutative triangle to "exact seq" ie using cones



- – To every triangle can assign long sequence
$A \in D \implies \ldots \to Hom(A,X) \to Hom(A,Y) \to Hom(A,C) \to Hom(A,X[1])$
$\to Hom(A,Y[1]) \to Hom(A,C[1]) \to \ldots$
which is long exact sequence of abelian groups

- Why exact? Suffices to show exact on one term (by rotation)

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & C \\ \vdots\uparrow & & \uparrow & & \uparrow \\ A & \xrightarrow{id} & A & \longrightarrow & 0 \end{array}$$
suppose have map $A \to Y$ going to 0 in C

$\Rightarrow$ extend by Tr3 to dotted arrow – so our long exact is exact!

- Consider all possible cones with given f:
claim they're all (noncanonically) isomorphic

$$\begin{array}{ccccccc} X & \xrightarrow{f} & X & \to & C & \to & X[1] \\ \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle id_Y} & & \vdots{\scriptstyle c} & & \downarrow \\ X & \xrightarrow{f} & X & \to & C' & \to & X[1] \end{array}$$

By five lemma show c is an isomorphism: can test on $\mathrm{Hom}(A,-)$ for all A, get isoms on Hom by five lemma $\Rightarrow$ c is an isomorphism

- Meaning of Tr 4: (note all axioms hold for homotopy category or derived category — standard properties of cones from Part I).
  - in derived category setting:

<u>Lemma</u>: $A \subset B \subset C$ abelian groups $\Rightarrow$ $C/B = (C/A)/(B/A)$ $\to$ gives Tr 4 for derived category!

any morphism of complexes can be represented by an injection: can lift to true morphism which is injective; any map between homotopy classes can be replaced by injection: mapping cylinders

For injective morphism of complexes, have

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \longrightarrow & Y/X \\ & & & \searrow & \uparrow \\ & & & & \mathrm{Cone}(i) \end{array}$$

quasi-isomorphism $\mathrm{Cone}(i) \to Y/X$ — not isomorphism in homotopy category unless sequence can be split ~~term~~ by term!

— So can take quotients for injections

$$\begin{array}{ccccc} & & & & Y/X \\ & & & \nearrow & \vdots \\ & & Y & & \vdots \\ & \nearrow & \downarrow & & \downarrow \\ X & \to & Z & \dashrightarrow & Z/X \\ & & & \ddots & \downarrow \\ & & & & Z/Y = Z/X / Y/X \end{array}$$

(Map:
$$\begin{tikzcd}
X \arrow[dr, "f"'] \arrow[rr, bend left, "h"] & & Z \arrow[dr] \arrow[rr, bend left] & & W \arrow[dr] \arrow[rr, bend left] & & \Sigma U \\
& Y \arrow[ur, "g"'] \arrow[dr] & & V \arrow[ur, dashed] \arrow[dr] & & \Sigma Y \arrow[ur] & \\
& & U \arrow[ur, dashed] \arrow[rr, bend right] & & \Sigma X \arrow[ur] & &
\end{tikzcd}$$

Tr4 axiom: existence of dotted lines making all 4 diagrams commute

- $Tr\ 1,2,4 \Rightarrow Tr\ 3$)

<u>Def</u> D triangulated category $\Rightarrow$ group $K_0(D)$
generators: objects of D. relations: for $X \to Y \to C$ exact triangle $[Y] = [X] + [C]$

($\Rightarrow [X[1]] = -[X]$ translation shifts signs)

$[X \oplus Y] = [X] + [Y]$

[Note: Axioms imply $X \to X \oplus Y \to Y \xrightarrow{0} X[1]$ is always distinguished : $\Rightarrow$ Class of triangles closed under $\oplus$]

Δ-category : "animation" of abelian groups - replace groups by richer structures on Δ-categories - "animation is inverse to K_0"

<u>Definition</u> · Triang functors
· Cohomology functors $H : D_{\text{triang}} \longrightarrow \mathcal{A}_{\text{abelian}}$:
sends distinguished triangles to exact sequences
$X \xrightarrow{f} Y \to C$ dist $\Rightarrow H(X) \to H(Y) \to H(C)$;
- hence get long exact sequence $\to H(X[i])$
$H^i(X) = H(X[i])$.

Can post compose with exact functors on abelian categories, still triang.

- Get map $H_*: K_0(D) \to K_0(\mathcal{A})$ <u>if</u> we assume $H^i(X) = 0$ almost all i
$[X] \longmapsto \Sigma(-1)^i H^i(X)$.

Typical example: <u>any</u> representable functor $Hom(A,-)$ sending $D \to Ab$ is a cohomology functor
(definition of $Ext^i(A,B) = Hom(A, B[i])$.)

Verdier's Localization

K : a triangulated category $\supset \mathcal{C}$ "strictly full" subcategory

Def: $\mathcal{C}$ is thick if • it is a triangulated subcategory
(closed under $\oplus$, $[1]$ & cones of morphisms
– assume $\mathcal{C}$ "strictly" full : $A \xrightarrow{\sim} C \in \mathcal{C} \Rightarrow A \in \mathcal{C}$)
and any direct summand of an object in $\mathcal{C}$ is in $\mathcal{C}$.

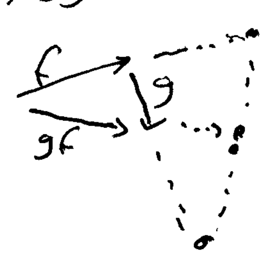
Typical examples: a. $F : K \to D$ any triangulated functor.
Then $\mathcal{C} = \operatorname{Ker} F$ is thick (objects s.t. $F(X)=0$)
b. $H : K \to \mathcal{A}$ a cohomology functor then
$\mathcal{C} :=$ H-acyclic objects is thick (all $H(X[i])=0$).

Def 1. $f : X \to Y$ in K is $\mathcal{C}$-quis ($\mathcal{C}$ thick) if $\operatorname{Cone}(f) \in \mathcal{C}$.
(e.g. if $\mathcal{C}$ comes from ex b this means f induces
isom on all cohomology groups.)
2. f is $\mathcal{C}$-negligible if it factors through an object of $\mathcal{C}$: $X \to C \in \mathcal{C} \to Y$

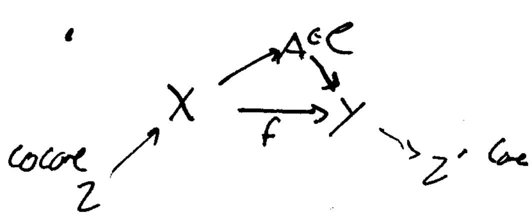
Lemma 1. $\mathcal{C}$-quis form a saturated multiplicative system
2. A morphism f is $\mathcal{C}$-negligible iff its composition
with a quis on left ($\Leftrightarrow$ on right) is zero.

(Saturated: $\xrightarrow{f}\xrightarrow{g}$ composable maps & any two of
f, g, gf are in our system $\Rightarrow$ so is third.)

Proof

• 

since cones of $\mathcal{C}$-morphisms are in $\mathcal{C}$ having any two $\mathcal{C}$-quis implies same for third.

• 

composition of two arrows in triangle is zero, use long exact sequence of triangles to get composition $=0$

So now can localize K by $\mathcal{E}$ just as in homotopy category! – kills every object of $\mathcal{E}$ ($0 \to \mathcal{E}$ quis)
In fact $\mathcal{E}$ exactly coincides with objects killed, & quotient is automatically triangulated.
(negligible morphisms $\leftrightarrow$ morphisms killed).

$\mathcal{E}$-Projective & Injective objects: $P \in K$ is $\mathcal{E}$-Proj if either of the following equiv properties holds:

i) $\mathrm{Hom}_K(P, -) = \mathrm{Hom}_{K/\mathcal{E}}(P, -)$
ii) $\mathrm{Hom}_K(P, A) = 0 \quad \forall A \in \mathcal{E}$
iii) $\mathrm{Hom}_K(P, -)$ sends $\mathcal{E}$-quis to isomorphisms.

All $\mathcal{E}$-Projs in K form a triangulated subcategory $\mathcal{P} \subset K$

Lemma If every object in K is $\mathcal{E}$-quis to a $\mathcal{E}$-proj object then $\mathcal{P} \subset K \xrightarrow{\sim} K/\mathcal{E}$

- the inverse functor $K/\mathcal{E} \to \mathcal{P}_{\subset K}$ is left adjoint to $K \to K/\mathcal{E}$

— i.e. Proj is "orthogonal complement" to $\mathcal{E}$!

Digression D a triang category, closed under countable direct sum
$X_0 \to X_1 \to X_2 \to \dots$ collection of maps (inductive system)

Homotopy direct limit $\underrightarrow{\mathrm{Holim}}\, X_i$: take $\oplus X_i$, has natural endomorphism t, $t|_{X_i} : X_i \to X_{i+1}$
$\underrightarrow{\mathrm{Holim}}\, X_i = \mathrm{Cone}(id - t : \oplus X_i \to \oplus X_i)$
- not canonically defined, only up to isomorphism.
Has maps $X_i \to \underrightarrow{\mathrm{Holim}}\{X_i\}$.

— In abelian category, replacing cone by cokernel, get just inductive limit $\varinjlim X_i$
(kernel is ~~___~~ zero for $Id - t$; upper triangular)

(countable direct sum: see functor - Hom from - is representable)

<u>Lemma</u> H any cohom. functor commuting with countable direct sum $\Rightarrow$ $H(\underrightarrow{\mathrm{Holim}}\, X_i) = \varinjlim H(X_i)$

<u>Proof</u> $\bigoplus X_i \longrightarrow \bigoplus X_i \longrightarrow \underrightarrow{\mathrm{Holim}}\, X_i$ dist triangle

- apply long exact sequence of cohomology

$$\begin{tikzcd} \bigoplus H(X_i) \arrow[r, "id - H(f)"'] & \bigoplus H(X_i) \arrow[r] \arrow[dr] & \text{cokernel is } \varinjlim H(X_i). \arrow[d] \\ & & H(\underrightarrow{\mathrm{Holim}}) \end{tikzcd}$$

$\dashrightarrow$ must check this is surjective (cokernel, injective)

To show surjectivity just need $\bigoplus H(X_i) \longrightarrow \bigoplus H(X_i)$ is injective (upper triangular again). ■

$\underrightarrow{\mathrm{Holim}}$ defined only up to noncan. isomorphism — unlike categorical direct limits.

<u>Construction of Projective Objects</u> Assume we have thick $\mathcal{C} \subset K$ (closed under arbitrary $\bigoplus$) s.t. $\mathcal{C}$ is defined by a cohomology functor which is representable & commutes with countable direct sums.

Then every object of K is $\mathcal{C}$-quis to a $\mathcal{C}$-proj object.

- e.g. in homotopy category of R-modules, H usual cohomology is $\mathrm{Hom}(R, -)$ - representable. This commutes with $\bigoplus$.
  e.g. ($\mathcal{C}$ = usual acyclic complexes)
- e.g. For a dg-assoc ring (unital) consider dg-modules
  - take homotopy category, derived category etc just as for usual R-modules, get Δ-categories.

  K = homotopy category, $\mathcal{C}$ = acyclic. H = cohomology, representable by our ring.

<u>Proof</u> $X \in K$ define projective $P \to X$ as $P = \underrightarrow{\mathrm{Holim}}\, P_i$

Set up
$$\begin{tikzcd} P_0 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[dl] & \dots \arrow[dll] \\ X \end{tikzcd}$$
$\Rightarrow$ get morphism from Holim: defined as cone (not nec unique!)

Want the following properties: a. P_i is $\mathcal{C}$-proj ($\Rightarrow P$ is $\mathcal{C}$-proj; direct sum of proj is proj)

b. $\forall i, a$ $H^a(P_i) \twoheadrightarrow H^a(X)$

c. $\mathrm{Ker}(H^a(P_i) \to H^a(X)) \longrightarrow H^a(P_{i+1})$ is <u>zero</u>.

b, c $\Rightarrow$ P is a resolution of X ($H^a(P) = H^a(X)$ in limit!)
<u>Construction</u> - by induction on i. Choose P_0 satisfying condition b : Let R be the object representing H, $H(X) = Hom_X(R, X)$. R is projective
- take sum of ∞ many copies of R & shifts to get condition b $P_0 = \bigoplus_{\text{"basis of" } H^*(X)} R[a]$

Next:

$$\begin{array}{l} Y \to P_0 \to X \\ \uparrow \\ Q \end{array}$$

Y cone of $P_0 \to X$ (making this distinguished)
Construct Q as the "P_0 for Y" - same construction
P_1 = cone $P_1' \to X$ comes from cone axioms.

$$\begin{array}{l} Y \to P_0 \to X \\ \uparrow \text{surj on } H^* \\ \bigoplus R[a] = Q \end{array}$$

big

Claim: this P_1 satisfies b, c : any class in $H^*(P_0)$ mapping to zero in X maps to zero in P_1'. - it must come from $H(Y)$, hence lifts to $H(Q)$, hence killed in $H(P_1')$
b ~~need not be free - just add lots~~ of copies of R.

[in category of R-mods / complexes all this is functorial using "the" cone rather than "a" cone - can replace $\underrightarrow{\text{Holim}}$ by $\underrightarrow{\text{Lim}}$. Find P_0 = sum of free modules projecting on X. add generators whose differential kills kernel on cohomology, keep adding ... $\blacksquare$

<u>Example</u> $\mathcal{A}$ abelian category having many projs <u>&</u> injectives (any X has $P \twoheadrightarrow X \hookrightarrow I$) & proj = injective : a <u>Frobenius category</u> - eg f.dim modules over a Frobenius algebra (f. some) - these are the algebras for which this is true.

[An algebra is Frobenius if it is <u>injective</u> as R-module]

e.g. group algebra of a finite group $k[G]$,
$\mathcal{A}$ = f.dim $k[G]$ modules (interesting when char $k \neq 0$!)
e.g. exterior algebra over a vector space, $\Lambda(V)$-modules

(Frob $\Longleftrightarrow$ dual to algebra R is projective. If this is ~~free~~ free: have linear "trace" functional generating R^*. R^* is mod-R, want it to be cyclic — like in $k[G]$ case .. i.e. need $\ell \in R^*$ s.t. $\langle a,b\rangle = \ell(ab)$ nondegenerate — not nec symmetric).

$\cancel{\mathcal{P}}\ \mathcal{I} \subset \mathcal{A}$ projectives ($\Longleftrightarrow$ injectives). Consider $\mathcal{A}/\cancel{\mathcal{P}}\ \mathcal{I}$: Same objects, morphisms are quotients by morphisms factoring through $\cancel{\mathcal{P}}\ \mathcal{I}$. Such morphisms form 2-sided ideal, can take quotient.
$\mathcal{A}/\mathcal{I}$ is additive — <u>NOT</u> abelian.
An object in $\mathcal{A}$ is zero in quotient $\Longleftrightarrow$ it's in $\mathcal{I}$ ($\mathcal{I}$ closed under direct summands).

1 <u>Claim</u> $\mathcal{A}/\mathcal{I}$ is naturally a triangulated category, s.t. for every short exact $0 \to A \to B \to C \to 0$, C is a cone of $A \to B$ in $\mathcal{A}/\mathcal{I}$ (i.e. give the triangles in $\mathcal{A}/\mathcal{I}$)
Translation: $X[1] = \mathrm{Cone}(X \to 0)$:
represent 0 by injective object. $X \to I \to I/X$, $X \to 0$, $I \to 0$
$X[-1]$: take $P \to X$ ....

2 • One way to see $\mathcal{A}/\mathcal{I}$: consider $D^b(\mathcal{A})$.
Contains subcategory $\mathcal{C}$ of fin. complexes of projectives ($\Leftrightarrow$ injectives) or its homotopy category, sits in $D^b(\mathcal{A})$: <u>thick</u>
$\Longrightarrow D^b(\mathcal{A})/\mathcal{C}$.
Claim: $D^b(\mathcal{A})/\mathcal{C}$ is equivalent to $\mathcal{A}/\mathcal{I}$

3 • Consider homotopy category of all acyclic complexes in $\mathcal{I}$ (all terms in $\mathcal{I}$). Claim: this is $\mathcal{A}/\mathcal{I}$.
... any such complex bounded either from above or below it's automatically ~~quis~~ homotopy eq. to zero!!

<u>Relations</u> $1 \longleftrightarrow 3$: given $A \in \mathcal{A}$ take proj & injective resolutions
$$\underbrace{P^\bullet \to A \to I^\bullet}_{\text{quis}}$$

Cone $(P^\cdot \to I^\cdot)$ is acyclic! get functor $\mathcal{A}$ to acyclic complexes of objects in $\mathcal{I}$

$$\ldots \to P^{-2} \to P^{-1} \to P^0 \qquad I^0 \to I^1 \to I^2 \to \ldots$$

Same for $2 \Leftrightarrow 3$: take resolutions on left $P^\cdot \to$ & right $\to I^\cdot$

e.g. for group ring of finite group : $Ext^i(k, V) = H^i(G, V)$
- Tate cohomology : ~~gen~~ glue together homology & cohomology to two-sided cohomology theory.

<u>Karoubian property</u> : idempotents (~~projectors~~) give direct sum decomposition -- true in abelian context - something intermediate between additive & abelian.
- Karoubian : every idempotent decomposes as projection onto direct summand.

<u>Lemma</u> : If a triangulated category D is closed under countable direct sums $\Rightarrow$ it is Karoubian

<u>Proof</u> $X \circlearrowleft P \quad P^2 = P$

Take $\underrightarrow{Holim}\,(X \xrightarrow{P} X \xrightarrow{P} X \to \ldots) = \mathrm{Im}\, P$

"Lazard lemma" : every flat R-module can be represented as $\varinjlim$ of free f.gen modules.
- represent X this way, then can describe $\mathrm{Im}\, P$ this way.

(Can use in context of coherent sheaves etc! "compact objects" in q-c sheaves where this property holds, so still get $\mathrm{Im}\, P$ etc.)

<u>Def</u> D triangulated category, $\{X_\alpha\}$ set of objects. X_α <u>generate</u> D if D is the smallest strictly full triangulated category containing X_α.
(contains all finite $\oplus$, shifts, cones, iterate..)

<u>Weaker version</u> : allow taking direct summands as well - Karoubian envelope : D = smallest thick subcategory containing X_α.

<u>Even weaker</u> ! For any $Y \in D$ $Y \neq 0$ one has $\mathrm{Hom}(X_\alpha, Y[a]) \neq 0$ some α, a
$\Longleftrightarrow$ right orthogonal complement to X_α is zero!

First def: if X_α (finite set) generate $\mathcal{D} \Rightarrow K_0(\mathcal{D})$ is finitely generated.
– NOT true if only generate in Karoubian sense.

Theorem (Kontsevich) X smooth quasi-proj variety, $D^b_{coh}(X)$ bounded derived category of coherent sheaves.
This is fin. generated in Karoubian sense!
[Not in first sense: K_0 can't be f.gen, contains Picard!]
More precisely, if $X \hookrightarrow \mathbb{P}^n$, then consider $\mathcal{O}_X, \mathcal{O}_X(1), \dots \mathcal{O}_X(n)$ – these generate.

Why? (i) $X = \mathbb{P}^n$ then $D^b_{coh}(X)$ gen by $\mathcal{O}_{\mathbb{P}^n} \dots \mathcal{O}_{\mathbb{P}^n}(n)$ in strong sense (Beilinson resolution) – stable category A/I for quiver algebra
(ii) Arbitrary X:
derived subcategory generated by these contains all $\mathcal{O}_X(i)$: Restrict complex from part (i) describing $\mathcal{O}_{\mathbb{P}^n}(i) \dots$
It remains to show all $\mathcal{O}_X(i)$ generate $D^b_{coh}(X)$
(iii) Suffices to show any coherent sheaf F on X belongs to Karoubian envelope.
Serre: $\{ \dots \to \overset{P^0}{\overset{\shortparallel}{\bigoplus \mathcal{O}(i)}} \to F \to 0 \} = P$
infinite left resolution with terms $\bigoplus \mathcal{O}(-i)$ different i's.
– doesn't help!
Use smoothness of X:
truncate at $N >> 0$ $\quad G \; \overset{T}{\overset{\shortparallel}{\{P^{-N} \to P^{-N+1} \to \dots \to P^0\}}} \; F$
has exactly two cohomology sheaves G, F.
$\longleftrightarrow$ element of $\mathrm{Ext}^{N+1}(F, G)$

X smooth $\Leftrightarrow$ finite cohomological dimension of D^b_{coh} (category)
So for $N >> 0$ this class is zero, so as object in derived category the truncated complex is isomorphic to a direct sum $T \simeq F \oplus G[N+1]$
So F = direct summand of T, which is a complex like we wanted! ∎

(really using that have fin loc free resolutions on smooth things –
& locally free things always are as direct summand of ~~stuff~~ complex bundle of our generators)

Thomason localization! X q-compact scheme, $U \subset X$ open (which is union of fin many affines (e.g. X Noetherian)) Given perfect complex on U (ie all cohomology sheaves coherent & finite Tor dimension $\iff$ on each open affine it is quism to a finite complex of vector bun's) $\implies$ can represent as direct summand of a complex, which can be extended to perfect complex on whole scheme.

Affine setting! can represent a vector bundle as direct summand of trivial vector bundle. This is nonaffine analog X q-compact

Claim: L perfect complex $\to$ can extend $L \oplus L[1]$
(ie obstruction lies in just K_0..)

A. Beilinson - Triangulated Categories III 11/26/01

Last time:
- If Δ category has countable $\oplus \Rightarrow$ Karoubian: contains images of idempotents
- Also finiteness properties ...

X object have $\bigoplus_{n=1}^{\infty} X$, shift is idempotent & cone is class of X - so in K theory $[X] = [\bigoplus^{\infty} X] - [\bigoplus X] = 0$ (?)

Thomason - Trobaugh Theorem: D a small triangulated category
Want to describe all subcategories $A \subset D$ (strictly full triangulated) s.t. every object $X \in D$ is a direct summand of an object of A : Smallest thick subcategory containing A is D.
e.g. over ring category of perfect complexes $\longleftrightarrow$ homotopy cat f.g. projective modules - all direct summands of free - generated by single object R.

Usually $D \neq A$: $K(A) \subset K(D)$ is just $\mathbb{Z}[R]$ in this sense ..

$$\{A \text{ as above}\} \rightleftarrows \{\text{Subgroups of } K_0(D)\}$$
$$A \longmapsto H_A = \text{Image of } K_0(A) \text{ in } K_0(D)$$
$$\mathrm{Ob}\, A_H = \{X : [X] \in H\} \longleftarrow H \subset K_0(D)$$

Theorem: These maps are mutually inverse, so $A \subset D$ as above $\longleftrightarrow$ subgroups of $K_0(D)$.

Proof: Obvious that A_H satisfies the conditions: definitely triangulated. Every object is direct summand of something in A_H: $X \hookrightarrow X \oplus X[1]$ which has zero class in $K_0(D)$ ($[X] + [X[1]] = 0 \in H \subset K_0(D)$).
Need to show that $A \subset A_{H_A}$ is everything
- i.e. objects X belong to A iff its class in $K_0(D)$ lies in $\mathrm{Im}(K_0(A))$. $\Longleftrightarrow$ image $[X] \in K_0(D)/\mathrm{Im}\, K_0(A)$ vanishes.
($\Rightarrow$ obvious, need $\Leftarrow$).

Replace $K_0(D)/\mathrm{Im}\, K_0(A)$ by another group for which statement is obvious. Look at stable equiv classes in D wrt adding A!
$X, Y \in D$ set $X \sim Y$ if $\exists A, B \in A$ s.t. $X \oplus A \cong Y \oplus B$.
- semigroup of objects in A acts on set of isom classes in D, take the quotient.

This gives a group G : from $\oplus$ of objects $\Rightarrow$ comm. semigroup. To find inverse X' s.t. $X \oplus X' \in \mathcal{A}$ $\longleftrightarrow$ class in K-group.

For small additive category have "stupid K_0-group" from $\oplus$ on isom classes - & this is stupid $K_0(\mathcal{D})$/stupid $K_0(\mathcal{A})$.

(For group G, have $[X] \in G$ vanishes $\Rightarrow X \in \mathcal{A}$! Can add to X something in A $[X \oplus A]_G \in \mathcal{A}$ - have $X \in A$: here we use that $\mathcal{A}$ is triangulated.)

Thus $[X]_G \in G$ vanish $\Leftrightarrow X \in \mathcal{A}$: $X \oplus B = A \Rightarrow X = \mathrm{Cone}(B \to A)$

b. $G \cong K_0(\mathcal{D})/K_0(\mathcal{A})$: First define a map $G \leftarrow K_0(\mathcal{D})/K_0(\mathcal{A})$ - characterized by $X \in \mathcal{D}$ $[X]_{K_0(\mathcal{D})/K_0(\mathcal{A})} \longmapsto [X]_G$ on generators. Certainly sends $X \in A$ to zero. So need to check relations coming from triangles!

$X \to Y \to Z$ triangle

[Generators in both sides same but RHS has more relations from Δ while LHS has only direct sum triangles...]

Choose objects complementary to X, Z in $\mathcal{D}$: $U, V \in \mathcal{D}$ s.t. $X \oplus U \in \mathcal{A} \ni Z \oplus V$ Now modify triangle:

$$\left\{\begin{array}{ccccc} X & \to & Y & \to & Z \\ \oplus & & \oplus & & \\ U & \xrightarrow{\mathrm{Id}} & U & & \\ & & \oplus & & \oplus \\ & & V & \xrightarrow{\mathrm{Id}} & V \\ \uparrow & & & & \uparrow \\ \mathcal{A} & & & & \mathcal{A} \end{array}\right\}$$

doesn't change classes

$\Rightarrow$ triangle with $X, Z \in \mathcal{A}$! Suffices to check relation $[Y]_G = [X]_G + [Z]_G$ for $Z, X \in \mathcal{A}$

- but then Y is also in $\mathcal{A}$ since $\mathcal{A}$ triangulated. So already part of relations we know in G !

~~In $K_0(\mathcal{D})/K_0(\mathcal{A})$ $[X+U] + [Z+V] = [Y+U+V]$~~

$[X] + [U] + [Z] + [V] = [Y] + [U+V]$

~~Properly $[Y]_G = [X]_G + [Z]_G \Leftrightarrow$~~

$[X]_G + [U]_G = 0 = [Z_G]_G + [V]_G$

$\Rightarrow$ $[Y]_G + [U]_G + [V]_G = 0$

$\Rightarrow$ $[X]_G + [Z]_G = [Y]_G$

Fact: $K_0(\mathcal{A}) \subset K_0(\mathcal{D})$ embedding – follows from theorem.. otherwise consider $H = \mathrm{Ker}(K_0(\mathcal{A}) \to K_0(\mathcal{D})) \subset K_0(\mathcal{A})$, apply theorem to it, get subcategory of $\mathcal{A}$, nontrivial
$\Rightarrow$ equivalent to a subcategory of $\mathcal{D}$, where it's zero
i.e. suppose $H \neq 0 \Rightarrow$ subcategory $\mathcal{B} \subset \mathcal{A}$, nonzero.
But $\mathcal{B}$ satisfies our property wrt $\mathcal{D}$ so is determined by the subgroup of $K_0(\mathcal{D})$ — but that is the zero subgroup. ■

TT Theorem: Inside projective scheme X have open U, ask to extend perfect complexes ... ?
[True for X qcompact & qseparated]

Theorem: U qcompact open $\subset X$ quasiprojective (for simplicity)
$F \in D^b_{perf}(U)$ [locally represented by finite complex of finitely generated projective modules]
$D^b_{perf}(U) \longleftarrow D^b_{perf}(X)$ restriction (neither injective nor surjective)
Then F can be extended to X (comes from $D^b_{perf}(X)$)
iff $[X] \in \mathrm{Im}\, K_0(X) \to K_0(U)$.

(ie classes of extendable complexes are those coming from X, everything is direct summand of extendables,
Don't need smoothness in argument from last time if restrict to perfect complexes: smooth means everything is perfect.)

X affine U quasiaffine:
Given complex F on U $\longrightarrow$ map to complex of its sections,

[Drinfeld] $X \overset{=\mathrm{Spec}\,A}{\text{affine}} \supset U$ open quasicompact (i.e. fin union of open affines $\leftrightarrow$ of principal open affines $D(f) = \{f(x) \neq 0\}$ $f \in A$
$F \in D^b_{perf}(U)$

1) F has an infinite left resolution by free finigen modules (need finiteness not perfection .. Noetherian case: need all cohomology sheaves coherent... Need: locally complex of global sections is quism to bounded above complex of free $\Leftrightarrow$ proj modules)
$$(\cdots \xrightarrow[d_1]{} \mathcal{O}_U^{m_1} \xrightarrow[d_0]{} \mathcal{O}_U^{m_0}) \xrightarrow{qis} F$$

1') Can make such a resolution so that differentials d_i extend to X (like sheaves $\mathcal{O}_U^{m_i}$) – complex exists on X

To see this assume F is module (not complex) with inf left resolution by free mod-ls. First find enough global sections of F generating as a sheaf : true on each affine open e.g. $D(f)$. After multiplying each such by power of f get something which extends to all X.

So we've shown that if sheaf on U is loc fin gen it's globally fin gen...

Careful - work with finite cover $U_i = D(f_i) \cap U$
Multiply on each by high enough powers of f_i

...

So we have $\mathcal{O}_U^{m_0} \twoheadrightarrow F$, look at kernel F'.
Has same property! locally finitely generated, get $\mathcal{O}_U^{m_1} \to \mathcal{O}_U^{m_0} \to F$.
Want differential to be extendable
$F' \subset \mathcal{O}_U^{m_0}$ need to show F' also extendable ....
$\Rightarrow$ get resolution. Same applies for complexes F.
$\cdot\ \mathcal{P} = (\cdots \to \mathcal{O}_X^{m_1} \xrightarrow{d} \mathcal{O}_X^{m_0})$ on X
$\overset{\text{``}}{\mathcal{P}^{-1}}$ $\overset{\text{``}}{\mathcal{P}^{0}}$

& $\mathcal{P}|_U \xrightarrow{qis} F$.

Now truncate : σ stupid truncation $\sigma_{\geq N} \mathcal{P} := (0 \to \mathcal{P}^{-N} \to \cdots \to \mathcal{P}^{0})$
$\sigma_{\geq N} \mathcal{P}|_U \to F$ has cocone$(\sigma_{\geq N}\mathcal{P}|_U \to F) = Z^{-N}[N]$
where $Z^{-N} = \mathrm{Ker}(\mathcal{P}^{-N} \to \mathcal{P}^{1-N})$

$N \gg 0$ Want to show that in derived category F is a direct summand of $\sigma_{\geq N} \mathcal{P}|_U$ - ie of complex extendable to X.
- We have a triangle $Z^{-N}[N] \to \sigma_{\geq N}\mathcal{P}|_U \to F \xrightarrow[\varphi]{} Z^{-N}[1+N] \to$.

Suffices to show $\varphi = 0$ (in fact equivalent to splitting)
in fact for N large enough no morphisms
$F \to Z^{-N}[1+N]$: $\mathrm{Hom}(F, Z^{-N}[1+N]) = 0$
$$\overset{\shortparallel}{R^0\Gamma}\,(R\mathcal{H}om(F, Z^{-N}[1+N]))$$
But locally F is a finite complex of free sheaves, Z^{-N} is just sheaf, length of $\mathcal{H}om$ is bounded above
- take $M >$ max length of a collection of finite complexes with $F|_{D(f_i)} \xrightarrow{qis}$ our complex of projectives

so $R\mathcal{H}om^{\cdot}$ is concentrated in degrees $[-M+1+N, 1+N]$
$R\Gamma$: get bounded complex with length bounded by
d of our open affines (Čech complex)
$\Rightarrow$ $[-M+1+N, \ 1+N+d]$
Choose N so large that $-M+1+N > 0$. $\blacksquare$

Drinfeld: Replace derived category of sheaves by homotopy category of injective complexes — naturally a dg category.
Want functors "essential equivalence" if give equivalence of homotopy categories. Well really must "localize" — need seesaws of such functors.
Problem with derived category: can't patch local to global

SGA 4 Vol 3 article by Deligne "Poincaré duality"
Champs de Picard strictement commutatifs
- look like complexes $[A_0] \to [A_1]$ in derived category. Each Picard stack does define such a thing but only recover Picard stack up to noncanonical isomorphism. For practical purposes use homotopy category of injective complexes.. can define $R\Gamma$ etc on these categories. Problem with essential equivalence of dg categories.

derived category of modules can be explicitly described using either injectives or projectives: Why use dg category?
Use seesaw of pairs $\{P^{\cdot} \xrightarrow{\text{qis}} I^{\cdot}\}$
$\{P^{\cdot}\} \leftarrow \quad \to \{I^{\cdot}\}$
— ad hoc solution

May: should work with model categories: two model structures
- one using projectives in privileged role, other uses injectives.
Have notion of Quillen equivalence of model categories

Can also use Dwyer-Kan seesaws: only use category & collection of "weak equivalences"
Σ-sd categories $\leftrightarrow$ simplicially enriched categories,
Any category $\overset{?}{\longleftrightarrow}$ dg enriched category.
BUT need to choose model structure..