JAS Langlands duality for character theories

Note Title 11/25/2007

(jt. with David Nadler)

Goal: describe a TFT framework for the representation theory of Lie groups.

Gauge theory vs representation theory:

2d G finite or compact

Explicit understanding of structure theory

4d Loop groups, geometric Langlands

S/Langlands duality; rich structure of line/Hecke operators

3d (noncompact) real & complex Lie groups

Happy medium! rich structure + prospect of fairly complete understanding!

$\mathcal{X}_G \simeq \mathcal{X}_{G^\vee}$

Baby example: 2d YM with finite gauge group.

G finite group $\rightsquigarrow$ 2d TFT Z_G

Group algebra: $\mathbb{C}G$ = measures on G under convolution. Forms a noncommutative Frobenius algebra with eval_1 as nondegenerate trace form.

$Z_G(S^1) = Z(\mathbb{C}G) = \mathbb{C}G^G = \mathbb{C}[\frac{G}{G}]$ class functions form commutative Frobenius algebra

$\rightsquigarrow$ determines $Z_G($  $)$ etc :

eg $Z_G(\Sigma_g) = \# \mathcal{M}_G(\Sigma) = \{\pi_1(G) \to \Sigma\}/\sim$

or fix conjugacy classes on circles...

– understood by Frobenius & Schur.

Open string version:

G-rep = $\mathbb{C}G$-mod is a CY category: have traces on endomorphisms of modules

$M = \oplus M_i^{\oplus n_i}$ $\quad$ End $M = \prod \mathrm{Mat}_{n_i}$

— looks like category of D-branes

Open-closed transitions 

$$\mathbb{C}G^G = Z(G\text{-rep}) \underset{\text{char}}{\overset{\text{act}}{\rightleftarrows}} \mathrm{End}\, M$$

char (1_i) = character of M_i.

Example: $M = \mathbb{C}[G(\mathbb{F}_q)/B(\mathbb{F}_q)]$
flag variety over a finite field

$$H_{fin} = \operatorname{End} M = \mathbb{C}[B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)]$$

$\mathbb{C}B_G$ Artin braid group acts $\quad W$

Generates sub-TFT: see only part of $Z_G(S^1)$ closed string states.

$G = G_{\mathbb{C}} [= G_{\mathbb{R}}]$ [real or] complex reductive group
Interesting class of reps: Harish-Chandra modules ($\supset$ unitary). Complicated category.
Harish-Chandra character:
distribution on $\frac{G}{G}$ satisfying strong (holonomic) system of diffeqs — analytic in generic directions, almost determined by these diffeqs — example of Lusztig's <u>character sheaves</u>.
Key idea: $\mathbb{C} \rightsquigarrow \mathcal{D}$
Ingredients: <u>group algebra</u> $\mathcal{D}G [= A(T^*G)]$
$\mathcal{D}$-modules (systems of diffeqs) on G
- substitutes for functions
$\{ f \rightsquigarrow [df] \in A(T^*G) \}$

<u>Theorem</u> (BZ-Francis-Nadler)

$$\underset{\text{monoidal}}{Z(\mathcal{D}G)} = \underset{\text{braided}}{\mathcal{D}\tfrac{G}{G}} \qquad \text{(Drinfeld double)}$$

- example of a general theory of centers of sheaves on derived stacks.

idea: 3d TFT Z_G:

	$\Sigma \times S^1$	Poincaré poly.
	Σ	$H^*_{dR}(\mathcal{M}_G(\Sigma))$
	S^1	$\mathcal{D}\frac{G}{G}$
	$\bullet$	G-cat $=$ $\mathcal{D}G$-mod

Focus on more understandable subtheory:

$$M = \mathcal{D}[G/B]$$

$$\mathcal{H}_{GB} = \mathrm{End}\, M = \mathcal{D}[B \backslash G / B]$$

Finite Hecke category: "$\nearrow B_G$" generated by Artin braid group

Replace $\mathcal{D}G$-mod by $\mathcal{H}$-mod.

Why do we care?

Beilinson-Bernstein: $M \simeq \mathfrak{g}$-rep

$\mathcal{H}$-mod $\ni \mathcal{D}(G_{\mathbb{R}} \backslash G/B) =$ HC-modules for $G_{\mathbb{R}}$ $= M_{G_{\mathbb{R}}}$

<u>Theorem</u> (BZ-Nadler)

$$\mathcal{Z}(\mathcal{H}_{G,B}) = \mathcal{C}har(G)$$

<u>character sheaves</u> for G

... in particular character sheaves are a braided (E_2) category.

[Caveat: really have theory over $\lambda \in \mathfrak{h}^*$: twisted $\mathcal{D}$-modules $\mathcal{D}(G/_{\lambda}B)$ - above is unipotent version, but theorem applies to all]

$\Rightarrow$ (?) 3d TFT χ_G:

$$\begin{array}{lcl} \Sigma \times S^1 & ? & P_+(\mathcal{M}_G(\Sigma)) \\ \Sigma & ? & H^*_{dR}(\mathcal{M}_G(\Sigma)) \\ S^1 & & \mathcal{C}har(G) \\ \bullet & & \mathcal{H}\text{-mod} \\ & & \mathcal{M}_{G_\mathbb{R}} = HC^{\omega}\text{-modules} \end{array}$$

<u>Main point</u> χ_G comes from dimensional reduction from (a subtheory of) $N=4$ SYM in GL twist $\rightsquigarrow$ get an S-duality for representation theory of $G, G_\mathbb{R}$.

Kapustin-Witten: conjectural equivalence of 4d TFT

$$A_G \simeq B_{G^\vee}$$

[N=4 SYM, GL twist $\Psi = 0$ $\Psi = \infty$]

$\Leftrightarrow$ geometric Langlands when compactified on a surface Σ.

Subtheory: tamely ramified surface operators of Gukov-Witten $\Longleftrightarrow$ affine Hecke category of Bezrukavnikov

Study modules $\mathcal{M} = \mathcal{D}(LG/I)$ affine flags

vs $\mathcal{M}^\vee = \mathcal{O}(T^*G^\vee/B^\vee)$

– correspond to introducing simplest ramification

– ... parabolic structure ... along a surface in 4d

$\longleftrightarrow$ at point of Σ

Theorem (Bezrukavnikov) $\operatorname{End} \mathcal{M} \simeq \operatorname{End} \mathcal{M}^\vee$ as algebras

$$\begin{array}{ccc} \operatorname{End} \mathcal{M} & \simeq & \operatorname{End} \mathcal{M}^\vee \\ \Big\downarrow{\scriptstyle S^1 \text{ localization}} & & \Big\downarrow{\scriptstyle S^1 \text{ localization}} \\ \mathcal{H}_{G,B} & \simeq & \mathcal{H}_{G^\vee,B^\vee} \end{array}$$

Theorem (BZ-Nadler) $\mathcal{H}_{G,B} \simeq \mathcal{H}_{G^\vee,B^\vee}$

$\rightsquigarrow$ Theorem (BZ-Nadler) " $\mathcal{X}_G \simeq \mathcal{X}_{G^\vee}$ "
Langlands duality for character theories.

More precisely have equivalence of
2-periodic versions of

$$
\begin{array}{lccc}
\text{monoidal} & \mathcal{D}(B\backslash G/B)_\chi & \longleftrightarrow & \mathcal{D}(B^\vee\backslash G^\vee/B^\vee)_\chi \\
 & \uparrow\downarrow & & \uparrow\downarrow \\
\text{braided} & \mathrm{Char}_\chi(G) & \longleftrightarrow & \mathrm{Char}_\chi(G^\vee)
\end{array}
$$

- Local Langlands program (ABV for K-groups, Soergel conj)

$$\bigoplus_{G_{\mathbb{R}}} M_{G/\mathbb{R}} \longleftrightarrow \bigoplus_{G^\vee_{\mathbb{R}}} M_{G^\vee_{\mathbb{R}}}$$

(can prove by TFT methods on K-group level, & for categories modulo Langlands base change / orientifolding)

- On surfaces: shall explain & prove conjectures of Hausel - Rodriguez Villegas

$$P_t(\mathcal{M}_G(\Sigma)) \longleftrightarrow P_t(\mathcal{M}_{G^\vee}(\Sigma))$$

& mixed Hodge (q-deformed) version